Chern Simons Theory and Quantum Hall.

Effective Theory of Laughlin States

- Electrons move in a 2-dimensional system with background magnetic field.
- Consider single-layer spin polarized quantum Hall
- Wish to write down an effective theory for Laughlin states
  that satisfy following properties:
- (1) Electromagnetic current \( J^m \) is conserved: \( \gamma^m J_m = 0 \).
- (2) Use a local Lagrangian: \( \text{Lagr} \[ \text{A} \] = \int d^2 x \ldots \) background field.
- (3) Only interested in long distance physics.
- Parity and time reversal are broken by the
  external magnetic field.
- Let \( A^m \) be background gauge field.

\[
\text{Lagr} \[ \text{A} \] = \int d^3 x \, J^m A_m + \ldots
\]

where

\[
\begin{align*}
J^1 &= e^{-i \vec{E} \cdot \vec{A}} (\mathcal{J}^{\vec{E}} - \mathcal{J}^{\vec{E}^*}) \\
J^2 &= e^{-i \vec{E} \cdot \vec{A}} (\mathcal{J}^{\vec{E}^*})
\end{align*}
\]

are the current and charge of the particles.

(1) \( \Rightarrow \) \( J^m = \frac{e^2}{2\pi h} \epsilon^{mnp} \partial_n \mathcal{A}_p \) Since no divergence

Choice of normalization becomes clear later.

An effective action that is consistent with (1)–(4) is

\[
\text{Lagr} \[ \text{A} \] = \frac{e^2}{2h} \int d^2 x \, \frac{1}{\delta x} \epsilon^{mnp} \text{And}_{mn} - \frac{m}{4\pi} \epsilon^{mnp} \text{And}_{mn} \ldots
\]

The \ldots includes irrelevant terms like the Maxwell term that
does not play role at large distances.

- Hall conductivity: Integrate over \( \text{A} \)
- P.O. m \( \text{F}_{mn} = \frac{1}{m} \epsilon_{mnp} \)
- Put back into action: \( \text{Lagr} = \frac{m}{2\pi h m} \int d^2 x \, \epsilon^{mnp} \text{And}_{mn} \)

\[
J^m = \frac{1}{\delta A_m} \frac{d \text{Lagr}}{d A_m} \Rightarrow \delta J^m = \frac{e^2}{2\pi h m} \epsilon^{nmp} \text{F}_{np}
\]

\[
\delta J^l = \frac{e^2}{2\pi h m} \epsilon^{lmp} \text{F}_{lp} \Rightarrow \delta x y = \frac{e^2}{2\pi h m}
\]
If we had added a Chern Simons term for $A$, it would simply give a integer contribution to $\xi_{\text{eq}}$. Excluding such a term corresponds to working in the lowest Landau level.

**Quasiholes/Particles:** We can introduce quasi particles by coupling a current to the emergent gauge field

$$\Delta S = \int d^2x \ A^\mu j_\mu$$

- For $A = 0$
  - P.O.M. for $A^\mu$:
    $$\frac{e^\mu}{\hbar c} \partial_\mu A^\nu = \frac{i}{2m} j^\nu$$
  - Charge $e$ under $A^\mu$.
- Static charge:
  $$j^i = j^0 = 0 \quad j^0 = e \delta^0(x)$$
  $$\Rightarrow \frac{i}{\hbar c} \delta_{i0} = \frac{e}{\hbar c} \delta (x)$$
- Chern-Simons attaches flux $\frac{\hbar c}{e m}$ to change $e$, which will change its statistics.
- The actual electric charge that couples to $A^\mu$ is $\frac{\hbar c}{e m} j^0 = \frac{e}{2m} \delta^0 (x)$
  - Fractional charge.
- Anyonic statistics:
  - Brand one quasi particle around the other.
  - Braiding statistics:
    $$q = e, \quad q = \frac{1}{2m}, \quad e: \frac{q_0}{q_0} = e \frac{2\pi}{\hbar}$$
  - Exchange statistics:
    $$e \frac{2\pi}{\hbar}$$

**Careful derivation:** $N$ particles at $\xi_{\text{eq}}(t)$

$$\int_0^1 \sum_{\alpha=1}^N \int d^2 x \ d^2 \xi \ d^2 \eta \ (\eta - \xi)$$

Solve O.D.M. 0 in Coulomb gauge $\nabla \cdot \vec{A} = 0$ with $\partial_0 \vec{A} = 0$.

and using 3d Green's function

$$\nabla^2 \log |x - y| = 2\pi \delta (x - y)$$

gives

$$\vec{A} = \sum_{\alpha=1}^N \int d^2 \xi \ d^2 \eta \ (\eta - \xi)$$

$$\vec{A}(\vec{x}) = \frac{\hbar c}{2m} \sum_{\alpha=1}^N \int \frac{d^2 \xi}{\xi - \vec{x}} \ \vec{A}(\vec{x} - \vec{\xi})$$

Aharonov-Bohm phase:

$$\exp \left( \frac{i e}{\hbar c} \oint \vec{A} \cdot d\vec{z} \right) = \exp \left( \frac{2\pi}{\hbar} \right)$$
Grand State degeneracy is a signature of topological order. Consider a 1-m-torus. Non-dynamical $\phi_0$ is a Lagrange multiplier that gives rise to a Gauss law constraint:

$$f_{i\alpha} = m_F f_{i\alpha}$$

Parametrize the space of solutions using Wilson loops, let $\gamma_1, \gamma_2$ be cycles on the torus:

$$W_i = \exp \left( \frac{\hbar}{i} \phi_0 \cdot \gamma_i \right)$$

Canonical commutation relations:

$$[\theta_i(\vec{x}), \mathbf{0}_i(\vec{x}')] = \frac{2\pi i}{m} \frac{\hbar}{2} \delta^{(2)}(\vec{x} - \vec{x}')$$

$$\mathbf{RCH} \Rightarrow W_1 W_2 = e^{\frac{2\pi i}{m}} W_2 W_1.$$ Smallest rep of this algebra has dimension $m^4$, which is grand-state deg.

For genus-$g$ surface, $g \cdot m = m^4$.

**Hierarchy and K-Matrix**

Haldane approach: View the gauge field $A_{\mu}$ as a fixed background in which the quasiparticles $\gamma^\mu$ move in. Then the quasiparticles themselves can form a quantum hall state of their own.

Repeat procedure and introduce a new $U(1)$ gauge field $\tilde{A}_\mu$.

$$\tilde{\mathcal{M}} = \sum_{\mu} \mathcal{E}^{\mu\nu} \tilde{A}_\nu$$

$$\Rightarrow \text{eff} = \int d^3x \left( \sum_{\mu} \mathcal{E}^{\mu\nu} A_\nu dA_\mu + \frac{\hbar}{m} \mathcal{E}^{\mu\nu} A_\nu \tilde{A}_\mu + \frac{\hbar}{m} \mathcal{E}^{\mu\nu} A_\nu \tilde{A}_\mu + \frac{\hbar}{m} \mathcal{E}^{\mu\nu} A_\nu \tilde{A}_\mu \right)$$

Hall conductivity: Integrate out $\tilde{A}_\mu$, then a

$$\Rightarrow \eta = \frac{1}{m - \frac{\hbar}{m}}$$

This new quantum hall state can have quasiparticles that can further condense into its own quasiparticle state.

In general, have $N$ emergent gauge fields $A_1, \ldots, A_N$.

$$S = \int d^3x \left( \sum_{\mu} \mathcal{E}^{\mu\nu} A_\nu \cdot dA_\mu + \frac{\hbar}{m} \mathcal{E}^{\mu\nu} A_\nu \cdot \tilde{A}_\mu + \frac{\hbar}{m} \mathcal{E}^{\mu\nu} A_\nu \cdot \tilde{A}_\mu \right)$$

where $K$ is the $K$-matrix, and $\mathbf{t}$ is the charge vector.

$$K = \begin{pmatrix} m & -1 & 0 \\ -1 & m & -1 \\ 0 & -1 & m \end{pmatrix} \quad \mathbf{t} = (1, 0, 0, \ldots)$$
Filling fraction: \( \nu = e^{-1} \)

Generic quasiparticle couples to the gauge fields as
\[ \lambda \rightarrow g_{\lambda} / e \]

Integrating out the gauge fields show that the quasiparticle carries \( \sum \lambda g_{\lambda} / e \) units of \( \lambda \)-flux.

\[ \Theta = \sum g_{0} \Phi_{0} = \lambda L \]

Electric charge:
\[ -e \lambda \]

\[ \text{Genus} \quad q \quad q \cdot s \quad 1 \text{dot} \quad L \]

\[ \text{Shift} \quad \text{&} \quad \text{Spin} \]

In semi-classical treatment, \( e^{-} \) in 2nd L.L has one more quantum of angular momentum than those in L.L. Would like to encode information about different angular momentum properties associated with different gauge potentials \( \lambda \).

Consider non-interacting (spinless) \( e^{-} \) on sphere.

Place magnetic monopole of strength \( G \) (either integer or half-integer) by Dirac quantization condition at center of sphere.

\[ \frac{1}{2} \frac{G}{4 \pi} \]

\[ \Phi = 4 \pi \frac{G}{e} \]

\[ N_{e} = \frac{\phi}{2e} = \frac{G}{2} \]

Single electron energy
\[ E = \frac{\hbar \omega}{G} \left[ \frac{1}{(4G)+1} \right] / G \]

Landau levels
\[ \lambda = G, G+1, G+2, \ldots \]

1\textsuperscript{st} level has degeneracy \( 2G+1 \)

\[ L \text{ Landau levels filled} \Rightarrow N_{e} = \sum_{G>0} \left[ \frac{2G+1}{4G+1} \right] = L N_{e} + L^{2} \]

\[ N_{e} = V \left( N_{e} - \frac{1}{2} \right) \quad \text{shift} \quad 1 \]

\[ C = V \text{ for a sphere} \]

Curved surface has a "gauge potential" \( \omega \) whose curl gives curvature of surface.
Adiabatic phase around loop
Variation of Lagrangian tells us
Electromagnetic properties

\[ L = \frac{1}{2} \epsilon \wedge \mathbf{A} \]
\[ e \wedge \mathbf{A} \]
Angular momentum properties

Curvature

\[ \epsilon \wedge \mathbf{A} \]
\[ e \wedge \mathbf{A} \]

\[ \frac{1}{2} \epsilon \wedge \mathbf{A} \]

No metric dependence \( \Rightarrow \) topological

Examples:
Sphere: Symmetric basis \( t^2 = 1 \)

\[ L = \frac{1}{2} \sum \epsilon (k^+) \] \( k^+ \)

Hierarchies:

\[ L = \frac{1}{2} \sum \frac{1}{2} \left[(k^-)^{-1}\right] \] \( k^- \)

Classification of Abelian Hall States

Label states with \( (\ell, l, s) \)

Change basis of gauge fields
\( k \rightarrow WkW^t \)
\( t \rightarrow Wt \)

Defined gauge fields so that flux on sphere is integer valued

\[ \frac{1}{2\pi} \int \mathbf{A} \in \mathbb{Z} \Rightarrow \mathbf{W} \in SU(N, \mathbb{Z}) \]

Edge Modes

Laughlin States: Grand state forms an incompressible disk (gapped in bulk).

Low energy excitations are deformations that change shape, not the area.

Wave-function approach: \( Y_{\ell}(\hat{\mathbf{r}}) \rightarrow Y_{\ell}(\hat{\mathbf{r}}) \prod_{i=1}^n \frac{e^{i e_i \hat{\mathbf{r}} \cdot \mathbf{a}_i}}{e^{e_i t} t} \)

\( e \) are fermions.

Powers of \( \ell \) \& angular momentum \& area.

\[ V_{\text{cont}} = \mathbf{W} \rightarrow \text{any mon. operation} \]
has unique ground state (Laughlin) + e:
with \( z(z_i) = 1 \)
\[ E_0 = \frac{w m}{2} \text{pures} 0.1 z. \]

Excited states: Basis of symm polynomials
\[ S_n(z_i) = \sum_i z_i^n. \]
State with \( S(z_i) = \prod_{n=1}^M S_n(z_i)^{d_n} \) has energy
\[ E = E_0 + w \sum_{n=1}^M d_n \]

Excitation energy \( \Delta E \)

<table>
<thead>
<tr>
<th>( d_i )</th>
<th>( \Delta E )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1 (d_i = 1)</td>
</tr>
<tr>
<td>2</td>
<td>2 (d_i = 2 or d_i = 1)</td>
</tr>
<tr>
<td>3</td>
<td>3 (d_i = 3 or d_i = 1, d_i = 2 mod 3 = 1)</td>
</tr>
</tbody>
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Partition of integer \( q \).

Chern Simons Approach
\[ \mathcal{L} = \frac{m}{4\pi} \int d^3x \epsilon^{abc} a_a d^b a_c \]
Variation gives boundary term
\[ \frac{m}{4\pi} \int d^2x (-a_\tau d_\tau + a_\tau d_\tau) \]
vanishes if we set \( \left(a_\tau - V^{\alpha x}\right)_{\left|_{\left|_{\left|_{y=0} \right.} \right.} \right. = 0 \)
introduced a parameter.

Gauge transformation.
\[ a_\mu \rightarrow a_\mu + \partial_\mu \lambda \]

Restrict to gauge transformations that vanish on boundary
\[ \lambda (x, 0) = 0 \]
This introduces dynamical degrees of freedom on the boundary
\( a_\mu - \partial_\mu \phi \)
on no longer gauge away \( a_\mu = \partial_\mu \phi \)

Fix: gauge. Extend \( a_\tau - V^{\alpha x} = 0 \) into bulk.

Change coordinates \( x' = t \rightarrow \xi' = t + V_1 \)
\[ \partial_{\xi'} = a_{\xi'} - V^{\alpha x} a_{\xi'}, a_{\xi'} = 0, a_{\xi'} = a_{\xi'} \]

Gauge fixing condition becomes \( a_1 = 0 \).

Regard \( \partial_\phi \) as constraint.
\[ f'' = 0 \Rightarrow a_1 = \partial_1 \phi \]
Pure gauge mode that is \( \phi \) invariant.
$L = \frac{m}{\hbar^2} \int d^3 x' \sum_{\alpha} \mathcal{J}^{\alpha} \cdot \partial' \mathbf{A}^\alpha$

$= \frac{m}{\hbar^2} \int d^3 x' \mathcal{J}^\alpha \cdot \partial' \mathbf{A}^\alpha \cdot \partial' \mathbf{A}^\alpha$

$= \frac{m}{\hbar^2} \int_{\phi=0} d^3 x' \mathcal{J} \cdot \partial' \mathbf{A}^\alpha$

$= \frac{m}{\hbar^2} \int_{\phi=0} d^3 x' \mathcal{J} \cdot \partial' \mathbf{A}^\alpha$

Define $P = \frac{1}{\hbar} \frac{\partial \phi}{\partial t} \Rightarrow \mathcal{J} \cdot \partial' \mathbf{A}^\alpha = -i \frac{\partial \phi}{\partial t} = \mathcal{J} \cdot \partial' \mathbf{A}^\alpha$

Chiral wave propagating at speed $V$

Consider coupling to background field. Original coupling

$A^\alpha = \sum_n \mathcal{A}_n \alpha_n \phi$ is not invariant under transformation

of $A^\alpha$ due to boundary, so instead work with a different coupling

$\frac{1}{\hbar} \int d^3 x \mathcal{J} \cdot \partial' \mathbf{A}^\alpha$

Set $A^\alpha = 0$ and turn on $A^\alpha$ dependent $A_L, A_R$

$S = \frac{1}{\hbar} \int d^3 x \mathcal{J} \cdot \partial' \mathbf{A}^\alpha$

$= \frac{1}{\hbar} \int d^3 x \mathcal{J} \cdot \partial' (A_L^\alpha - A_R^\alpha)$

$= \frac{1}{\hbar} \int d^3 x \mathcal{J} \cdot \partial' (A_L^\alpha - A_R^\alpha)$

Integrate first term by parts gives

$\mathcal{J} \cdot \partial' = D \phi - V \mathcal{J} \cdot \partial \phi = 0 \Rightarrow \mathcal{J} \cdot \partial \phi$ is boundary charge density

$\Rightarrow -\mathcal{J} \cdot \partial \phi$ is current.

Wave fluctuations where boundary deviates from circular droplet. If height $h(x, \phi, \psi)$

so $\phi$ has period $2\pi l$ (compact boson). This is because $\mathcal{J} \cdot \partial \phi$

Actually $A^\alpha \cdot \partial' \mathbf{A}^\alpha$ is $\delta^4 (\mathbf{x} - \mathbf{x}')$ with $\mathcal{J} = \mathcal{J} \cdot \partial \phi$

Hence $\delta^4 (\mathbf{x} - \mathbf{x}')$ is the density at the in state

$\phi$ is compact.

Can write $\rho = \frac{2 m \phi}{\hbar^2}$

For a quantum Hall droplet, $\rho \in [0, L)$,

$\rho = \frac{\rho}{\rho_0} \int_0^L d\rho = \frac{1}{\rho_0} \int_0^L d\rho = \rho \Rightarrow \rho$ is quantized.
Chiral Boson: Consider \( \Theta \)-lines a circle with circumference \( L = 2\pi \sqrt{\text{m} c^2} \), \( \sigma \in [0, L) \).

To quantize theory, go to Fock space:

\[
\phi (\sigma, t) = \frac{1}{\sqrt{I}} \sum_{n=0}^{\infty} \phi_n(\sigma) e^{\frac{\text{i} 2\pi n t}{L}}
\]

\[
\mathbf{p}(\sigma, t) = \frac{1}{\sqrt{I}} \sum_{n=0}^{\infty} \mathbf{p}_n(\sigma) e^{\frac{\text{i} 2\pi n t}{L}}
\]

we have \( \mathbf{p}_n = \frac{i\text{k}_n}{\sqrt{\text{m}}} \phi_n \) where \( \text{k}_n = \frac{2\pi n}{L} \).

- If \( \phi, \mathbf{p} \) real \( \Rightarrow \phi^* = \phi, \mathbf{p}^* = -\mathbf{p} \)
- \( \mathbf{p}_0 = 0 \) because \( \phi_0 \) decouples, so \( \sigma = \pi \) and set \( \phi_0 = 0 \).

\[
S = \frac{\hbar}{4\pi} \int dt \sum_{n=0}^{\infty} \text{ik}_n \phi_n \phi_{-n} + \text{v} \text{k}_n \text{k}_{-n} \phi_n \phi_{-n}
\]

Consider \( \phi_n \) with \( n > 0 \) to be the coordinates and their momenta is proportional to \( \phi_{-n} \).

Commutation relations are the \( \mathcal{U}(1) \) Kac-Moody Algebra

\[
[\phi_n, \phi_m] = \frac{2\pi}{\hbar} \frac{1}{\text{k}_n} \delta_{nm}
\]

\[
[\mathbf{p}_n, \phi_m] = \frac{\text{i}}{\hbar} \delta_{nm} \mathbf{d}
\]

\[
[\mathbf{p}_n, \mathbf{p}_m] = -\frac{\text{i}}{\hbar \text{m}} \delta_{nm} \mathbf{d}
\]

Fock space transforms back to \( \sigma \) equal time commutation relations.

\[
[\phi(\sigma), \phi(\sigma') = \frac{\pi}{\hbar} \text{v} \text{g} \pi (\sigma - \sigma') \]

\[
[\mathbf{p}(\sigma), \phi(\sigma') = \frac{\text{i} \hbar}{\text{m}} \delta (\sigma - \sigma') \]

\[
[\mathbf{p}(\sigma), \mathbf{p}(\sigma') = -\frac{\text{i}}{\hbar \text{m}} \delta (\sigma - \sigma') \]

Hamiltonian:

\[
H = \frac{\text{mv}}{2\hbar} \sum_{n=0}^{\infty} \text{k}_n^2 \phi_n \phi_{-n} + \text{J} \sum_{n=0}^{\infty} \mathbf{p}_n \mathbf{p}_{-n}
\]

where we normal ordered the operators. This is a bunch of decoupled oscillators.

Time dependence \( \dot{\phi}_n = \frac{\text{i}}{\hbar} [H, \phi_n] = \text{i} \text{v} \text{k}_n \mathbf{p}_n \)

States:

- Ground state \( |0\rangle \) with \( \mathbf{p}_n |0\rangle = 0 \) for \( n > 0 \)

- Excited states: \( |n\rangle = \prod_{n=0}^{\infty} \mathbf{p}_n |0\rangle = >H|n\rangle \approx \sum_{n=0}^{\infty} \frac{\text{J} \text{k}_n^2}{2\hbar} n |n\rangle + |0\rangle \)

Recover the spectrum and degeneracy of the Laughlin state excitations obtained earlier.
Because $\phi$ is periodic, we can construct a single valued operator as follows:

$$\mathcal{F} = e^{i\phi}$$

This produces a unit charge while $\mathcal{F}$ removes a unit charge.

BC: $\Rightarrow -\mathcal{F}(\sigma)\mathcal{F}(\sigma') = e^{-m^2[\phi(\sigma),\phi(\sigma')]\mathcal{F}(\sigma')\mathcal{F}(\sigma)}$

For $\sigma = 0$,

$$[\phi(\sigma),\mathcal{F}(\sigma')] = 0, \quad m \text{ even} \Rightarrow \mathcal{F} \text{ is bosonic}$$

$$[\mathcal{F}(\sigma),\mathcal{F}(\sigma')] = 0, \quad m \text{ odd} \Rightarrow \mathcal{F} \text{ is fermionic}$$

This matches up with Laughlin wave function, which is bosonic for even $m$ and fermionic for odd $m$.

Also have $\mathcal{F}_q = e^{i\phi}$

$$[\mathcal{F}(\sigma),\mathcal{F}_q(\sigma')] = -\frac{1}{m} \mathcal{F}_q(\sigma') \mathcal{F}_q(\sigma)$$

Create particles with charge $\pm 1/m$.

Statistical phase $e^{i\pi/m}$, which depends on whether we do a clockwise or anti-clockwise braiding.

Propagator: Deriving the propagator directly from the action requires a tricky contour integral. Let us just take the left moving part of the propagator for a normal boson.

$$S = \frac{m}{8\pi} [J^2 \times dJ \cdot d\Phi]$$

$$\langle \Phi(x,t)\Phi(0,0) \rangle = -\frac{i}{m} \log(v^2 \cdot x^2)$$

where correlation fns are time ordered and UV cutoff in log has been dropped.

$$\langle \Phi(x+vt)\Phi(0,0) \rangle = -\frac{i}{m} \log(x+vt) + \text{const.}$$

Careful derivation indeed gives

$$\langle \phi(x+vt)\phi(0,0) \rangle = -\frac{i}{m} \log(x+vt) + \text{const.}$$

Differentiating gives

$$\langle p(x+vt)\phi(0) \rangle = -\frac{1}{(x+vt)^2} \frac{1}{m} \frac{1}{(x+vt)^2}$$
To get the electron propagator, use the following result:

For some linear combination of creation and annihilation operators, \( A = \alpha \partial + \beta \partial^* \),

\[ \langle 0 | e^{A_t} \cdots e^{A_0} | 0 \rangle = \exp \left( \sum_{n=1}^{\infty} \frac{A_n}{n!} \right) \]

Since \( \phi \) is a collection of harmonic oscillators,

\[ G_{\Phi(t)} = \langle \Phi(t,1) \Phi(0,0) | \rangle \exp \left( m^2 \phi(1,1) \phi(0,0) \right) \]

When \( m \neq 1 \), this deviates from \( G_{\text{Luttinger}} \)

\[ \Rightarrow G_{\Phi(t)} \approx \frac{1}{(2\pi)^{1/2}} \]

Electrons on the edge of FQH are strongly correlated and cannot be described by Fermi liquid theory.

Call it chiral Luttinger liquid.

\( \Phi(t) \) has dimension \( m' \), so \( \Phi \) has dimension \( m' \).

Since \( e^t \) are fermions, \( m \) must be odd (fermionic statistics).

As Hamiltonian is varied continuously, the exponent has to remain invariant, and can only change when the bulk undergoes a phase transition, so the dimension of the operator can characterize the bulk phase.

**Bulk boundary correspondence**

- Set \( n = 1 \).
- Wick rotate to Euclidean space

\[ \bar{w} = \frac{z \bar{z}}{2} \]

Map to plane

\[ \bar{z} = e^{i\bar{w}} \]

Which are single valued.

- Chiral boson depends on \( \bar{z}+i\bar{w} \)

\[ \Rightarrow \phi = \phi(\bar{z}) \text{ is holomorphic.} \]

In the plasma analogy at Laughlin states, had particles of charge \( q = -m \) moving around a background charge density \( p_0 \)

\[ p_0 = \frac{2\pi \lambda_s}{\sqrt{MN}} \]

Now, consider the correlation function

\[ \langle e^{im\phi(z)}, \ldots, e^{im\phi(z')} \rangle \]

Where \( \Omega \) is the disk shaped region of radius \( R \).

Our Lagrangian is invariant under shift \( \phi \rightarrow \phi + \epsilon \) (since \( H \) contains derivatives of \( \phi \)).

- Our Lagrangian is invariant under shift \( \phi \rightarrow \phi + \epsilon \)

\[ m \Omega = -p_0 \int d^2 \bar{z} = \frac{2\pi \lambda_s}{\sqrt{MN}} \]

Choose the radius of the quantum Hall droplet.
Applying our previous formula, we get:

\[
\exp \left( -\frac{2\pi i}{\ell} \int_{\mathbb{C}} m^2 \left( \phi(z) \phi(z') \right) \right) - \frac{2}{\ell} m \pi \int d^2 z' \left( \phi(z) \phi(z') \right) - \frac{2}{\ell} m \pi \log |z - z'| \left( \phi(z) \phi(z') \right)
\]

\[
= \prod_{i=1}^{n} (z_i - z_i')^m \exp \left( \frac{2}{\ell} m \pi \int d^2 z' \log |z - z'| \right)
\]

\[
\approx \prod_{i=1}^{n} (z_i - z_i')^m e^{-\frac{2}{\ell} m \pi |z_i|^2} \text{ neglecting terms in } \frac{|z_i|^2}{\ell}
\]

Can also include quark fields:

\[
\left< \prod_{i=1}^{n} \phi(z_i) \prod_{i=1}^{m} \phi(z_i') \right>
\]

\[
= \prod_{a,b} (a_i - a_j)^m \prod_{i=1}^{n} (z_i - z_i') \prod_{i=1}^{m} (z_i - z_i) \prod_{i=1}^{n} (z_i - z_i') e^{-\frac{2}{\ell} m \pi |z_i|^2} \frac{1}{|z_i|^2}
\]

What is the connection between bulk and boundary?

**Chem Simons Wave-Function**

Consider $C_{12}$ on $\mathbb{R}^2$.

1. $a_0 = 0$ gauge, comm rel. $[a_i(z), a_j(w)] = \frac{2\pi i}{\ell} \delta(z - w)$
2. Complex coordinates $z = x + iy$, $a^2 = a^0 + ia^1$

\[
[a_z(z, \bar{z}), a_{\bar{z}}(w, \bar{w})] = \frac{2\pi i}{\ell} \delta(z - w)
\]

Write down a "Schrodinger" equation. Need a wavefunction in position space.

**Holomorphic Quantization:** Let $a_z$ be position, $a_{\bar{z}}$ be momentum.

\[
\mathcal{H} a_z(z, \bar{z}) = \frac{\delta^2}{\ell^2} a_z(z, \bar{z})
\]

$H_{CS} = 0$. Impose constraint $f_z = 0$ on operator eqn on $\mathcal{H}$.

\[
\left( \frac{\partial}{\partial a_z} - \frac{m}{\ell^2} \frac{\partial^2}{\partial z a_{\bar{z}}} \right) \mathcal{H} a_z(z, \bar{z}) = 0
\]

**Chiral Boson Partition Function**

1. Couple current in $F_1$ to a background gauge field.

\[
\mathcal{S} = \int d^2 z \left( \frac{m}{\ell^2} \partial^2 + \phi \right) \phi
\]

2. No $\frac{\partial\phi}{\partial \bar{a}}$ because $\phi$ is holomorphic.

\[
\mathcal{P} = \frac{1}{\ell^2} \frac{\partial}{\partial a_\bar{z}}
\]

\[
\mathcal{P} = \frac{\partial}{\partial a_\bar{z}} \phi = \frac{\partial}{\partial a_\bar{z}} \frac{\partial}{\partial z} \phi
\]

\[
\mathcal{P} = \frac{\partial}{\partial a_\bar{z}} \phi = \frac{\partial}{\partial a_\bar{z}} \phi
\]

Couple charge to background gauge field

\[
\mathcal{S} = \int d^2 z \left( \frac{m}{\ell^2} \partial^2 + \partial_z \phi \right) \phi
\]

\[
\frac{\partial}{\partial a_\bar{z}} \phi = \partial_z \phi = \partial_z \phi
\]
Extra term is $A z$. Promoted $D z$ into $D_{\pi}$ but not $D_{\psi}$ because it's a chiral boson.

\[ \text{e.o.m.} \quad D_{\pi} D_{\psi} \phi = \frac{1}{2} D z A_{z} \]

Charge $p$ no longer conserved. Anomaly.

Consider partition function, which is a function of background field.

\[ Z[A_{\pi}] = \int \mathcal{D} \phi \ e^{-S[A,\phi]} \]

\[ = \int \mathcal{D} \phi \ e^{-S \left[ D_{\pi} \left( \frac{m}{\sqrt{\lambda}} \ D z \phi \right) - \frac{m}{4 \pi} D_{\pi} A_{z} \right]} \]

\[ = \frac{m}{\sqrt{\lambda}} < D_{\pi} D_{\psi} \phi - \frac{1}{2} D z A_{\pi} > \]

\[ = 0 \quad \text{by e.o.m.} \]

Since $\Phi(A_{\pi})$ and $Z[A_{\pi}]$ solve the same equation,

\[ \Phi(A_{\pi}) = Z[A_{\pi}] \]

generates boundary corr. for bulk CC wavefunction.

References
1) D. Tong, Lectures on the Quantum Hall Effect (2016)
2) X.G. Wen, Topological Orders and Edge Excitations in FQH States, Advances in Physics, 44, 405 (1995)