Last week we saw tensor networks like MPS, PEPs. These mimic the geometry of the system being modeled, e.g. for spin chain

and work well for gapped systems: $S \leq 2 \log \xi$ (size of bond being independent of size).

For gapless systems, a better alternative is the Multiscale Entanglement Renormalization Ansatz (MERA).

Basic idea is to add another dimension to the network, corresponding to RG flow. This should sound a bit like the radical discrete in AdS/CFT, as we will make more precise later.

**Barriers of MERA**

We want to implement RG in real space, corresponding to a coarse-graining transformation. In more detail, take a 1D lattice system, implement coarse-graining with an isometry,

$$W: V_s \rightarrow V_{s'} \otimes V_{s''}, \quad W^T W = \mathbb{1}_{V_s}, \quad W W^T = P$$

states mapped $\psi \rightarrow \psi' \oplus \psi''$.

operator, e.g. Hamiltonian is mapped as

$$H' = W^T H W.$$

How faithful is this transformation? Look at the density matrix

$$C = \sum_{n}^{\infty} \rho^n = |Y \times Y| \sum_{\alpha} \rho_{\alpha} (Y_{\alpha} \times Y_{\alpha})^1 = \sum_{\alpha} p_{\alpha} |Y_{\alpha} \times Y_{\alpha}|^1, \quad \sum_{\alpha} p_{\alpha} = 1.$$

Then we should choose $W$ s.t.

$$P = W W^T = \sum_{\alpha} |Y_{\alpha} \times Y_{\alpha}|^1$$

In practice, we can neglect $\alpha$ with smallest $p_{\alpha}$, accepting
\[ 1 - \sum_{\alpha=1}^{\infty} \rho_{\alpha} \leq \varepsilon \ll 1, \]

\( \chi \) is a measure of entanglement between \( S_1 S_2 \) and the rest of the system:

\[ S = -\sum_{\alpha=1}^{\infty} \rho_{\alpha} \log \rho_{\alpha} \leq \log \chi, \text{ saturated for } \rho_{\alpha} = 1/\chi. \]

The \textit{Deconfinement of DMRG}.

For critical systems, if we repeat this many times, we need larger and larger \( \chi \) for accuracy. This reflects a failure of the simple coarse-graining: it is affected by entanglement at coarse scales smaller than the graining we have, if the regions are at the boundary.

For example: Take the true state \( n, s \), in a state

\[ \frac{|M_{n}, M_{s}, + K_{n}, K_{s}\rangle}{\sqrt{2}}. \]

If coarse-graining does \( |s'\rangle \), don't contribute to entanglement? \( \langle\rangle_{n, s} \text{ vs } \langle\rangle_{n, s}, \text{ removed} \), i.e. \( |s\rangle \) in a pure state.

\[ \text{If instead they belong to different blocks,} \]

entanglement never removed!

So this retains short-range features; not a proper RG flow.

(Still works for 1D gapped systems, DMRG)
We can modify the procedure by adding disentanglers, 
\[ u: \mathbb{C}^d \rightarrow \mathbb{C}^d, \quad u u^* = u^* u = 1, \]
along the boundary of \( d \) blocks before we cone-gain them, 
such that it removes the entanglement between the boundary.

Pictorially:

Again, map \[ H = \tilde{W}^+ u^* \tilde{H} u W^+ \]
The density matrix now has a smaller rank, that in 
particular does not increase, even for 1D critical systems, as 
the procedure goes on.

Note, we can swap operators up or down the \( \mathbb{C}^d \) ladder.

Our final tensor network MERA is the collection of 
all disentanglers and isometries. For a translation-invariant system, 
all \( u, \tilde{u} \) come for each layer, depending on \( O(N^d) \) parameters. 
Since there are \( O(\log N) \) layers, whole thing gives 
\( O(N^d \log N) \).

Also true for 2D systems in the form \( O(N^2 \log N) \).

In practice, this is what is optimized to find states of 
given Hamiltonian.
On a critical system, since it is scale-invariant, the picture is improved: disentanglers and invariants should have the same form for all layers, only two terms \( t_i \) to optimize on.

This method was used with \( K = 16 \) to find scaling dimension of operator of 1D Ising chain. Accurate up to 0.527.

- The advantage of MERA is it yields meaning of scale invariant picture on a lattice, without taking continuum limit.
- MERA can be seen as a quantum circuit. We will see more of this viewpoint next week.
- Support of an operator does not grow.

Behaviour of \( S_A \) entanglement entropy:

\[ S_A \] for a block of lattice site, need the density matrix. How to compute it?

We get entropy from tracing out that occurs when one cut a bond. So we get an upper limit for the entanglement entropy interpretable as \( S_A \) \( \leq N \log X \) bonds cut in a \( (d+1) \)-dimensional space (sound familiar?)
Can we express this length in terms of a metric?

For critical systems, each layer is identical.

How many bonds do we break at each layer?

10: 2

Layer needed: \( \log_2(L) \)

\[ S_t \leq 2 \log_2L \log X \]

PROD:

\[ \left( \frac{L}{2^D} \right)^{d-1} \]

Layer: \( \log_d(L) \)

\[ S_t \leq \log_2 L \log_2 \left( \frac{L^d}{2^{D(d-1)+2}} \right) \]

Interpret this as an area:

for a labelling layer,

\[ d\sigma^2 = R^2 \left( d\theta^2 + e^{-2\frac{r}{L}} dx^i dx_i \right) \]

\[ = R^2 \left( \frac{d\gamma^2}{\gamma^2} + \left( \frac{dx^i dx_i}{\gamma^2} \right) \right) \]

\[ \Rightarrow \text{spatial part of AdS metric!} \text{ (state at fixed time)} \]

for this part, count bonds you would cut if you closed surface on that layer.

Note: original system is at small \( r \), small \( \gamma \).

Boundary of AdS.

When \( \gamma \) \rightarrow 0:

\[ \frac{d\gamma}{\gamma} \approx \frac{d\gamma^2}{2\gamma^2} \]

RT backwards: start from entanglement, build geometry for structure of entanglement in the tensor network.
If correlation length is finite, entanglement is removed after finite number of steps. Space ends.

If we start MERA with a thermal state, at some point in the process we find a density matrix for a fully mixed state, proportional to $\rho$. This looks like a temperature $\sim$ horizon of a BH?

Where do things like large $N$ enter? For CFT, need fields to be maximally entangled. If not, this changes, and we need the geometric interpretation to change ...

Use this to move away from large $N$, AdS?

How do we use this to find time component?

Quantum circuit...

Ref:

cond-mat/0512165
quant-ph/0610099
0912.1651
1106.1082
0905.1317

$\text{MERA by Guifre Vidal}$
$\text{Review by "" ""}$
$\text{Eventide, Vidal}$
$\text{MERAS geometry}$