Kinetic Theory of Chiral Fermions

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4. Collisions and H-Theorem.

Refs:
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0. Motivation.

Description of chiral fermion ensemble has wide applications → quarks in heavy ion collision
→ Neutron Star? Early Universe?
→ excitations in Weyl semimetal
→ ...

Existed theories: QFT: more useful in microscopic regime
Hydrodynamics: near equilibrium.

Kinetic Theory fills the gap in between:

$$\text{QFT} \xrightarrow[\text{weak ext. field}]{\text{Macroscopic}} \text{Kinetic theory} \xrightarrow{\text{near equilibrium}} \text{hydrodynamics.}$$

The kinetic theory should have two features:
- Chiral anomaly must be built-in.
- Observables (current etc.), collisions, entropy, relaxation etc. must be Lorentz invariant ⟶ has very non-trivial implications.

1. Classical Action, derivation from path integral.

It is well-known that the Berry curvature for right-handed Weyl fermion is: $\epsilon_{ijk} b^k = \epsilon_{ijk}\, s\hbar \frac{p^k}{|\vec{p}|^3}$, $s = +\frac{1}{2}$

So that the Berry phase for a closed loop in $\hat{p}$ is equal to $\frac{1}{2} \times$ (solid angle enclosed).

Thus, we expect the action to be:

$$S = \int \left[p_i\, dx^i - a^i_{(p)}\, dp_i - |\vec{p}|\, dt \right] \quad \text{(free particle)}$$

where $\partial_{p_i} a_j - \partial_{p_j} a_i = \epsilon^{ijk} b_k$.

Now we derive this from the quantum theory.

The quantum propagator is:

$$G(0,T) = \mathcal{T} \exp\left[\frac{-i}{\hbar}\int_0^T dt\, \left(\hat{p}_i - A_i(t,\hat{x})\right)\sigma^i + A^0(t,\hat{x})\right]$$

$$= \prod_t \exp\left(-\frac{i}{\hbar} dt\, \hat{p}_i \sigma^i\right) \exp\left(\frac{i}{\hbar} dt\, (A_i \sigma^i - A^0)\right)$$

Insert $\int d^3p(t)\, d^3x(t) \quad |p(t)\rangle\langle p(t)|x(t)\rangle\langle x(t)| \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$G = \prod_t \int \frac{d^3p(t)\, d^3x(t)}{(2\pi\hbar)^3} \exp\left[-\frac{i}{\hbar}\left(x^i_{(t)}\left(p_{(t)} - p_{(t-dt)}\right)_i + A^0_{(t,x(t))} dt\right)\right]$$

$$\times \exp\left(-\frac{i}{\hbar} dt\, \sigma^i p_i(t)\right) \cdot \exp\left(\frac{i}{\hbar} dt\, \sigma^i A_i(t,x_t)\right)$$

Decompose $\sigma^i p_i = U(p) \begin{pmatrix} |p| & 0 \\ 0 & -|p| \end{pmatrix} U^\dagger(p)$ where:

$$U(p) = \begin{pmatrix} u_+(p) & u_-(p) \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2} & -e^{-i\phi}\sin\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$ (Dirac string placed along $\theta = \pi$)

Thus:
$$G = \prod_t \int \frac{d^3p(t)\, d^3x(t)}{(2\pi\hbar)^3} \exp\left[\frac{i}{\hbar}\begin{pmatrix} p_i dx^i - |p|dt - A^0 dt & 0 \\ 0 & p_i dx^i - (-|p|)dt + A^0 dt \end{pmatrix}\right](t)$$

$$\times\, U^\dagger(p(t)) \exp\left(\frac{i}{\hbar} dt\, \sigma^i A_i(t,x_t)\right) U(p(t-dt))$$

When $A_\mu = 0$, $U^\dagger(p(t)) U(p(t-dt)) \simeq 1 - \frac{i}{\hbar} dp^i \left(\underbrace{U^\dagger(-i\hbar\partial_{p^i}) U}_{\text{matrix of Berry connection}}\right)(p(t))$

$$\begin{pmatrix} a^i(p(t)) & \cdots \\ \cdots & -a^i(p(t)) \end{pmatrix}$$

The $(\cdots)$ terms are odd in $\vec{p}$, and vanish under integral. Corresponds to NO hopping between particle and anti-particle mode.

$$G = \prod_t \int \frac{d^3p(t)\, d^3x(t)}{(2\pi\hbar)^3} \begin{pmatrix} \exp\left[\frac{i}{\hbar}\left(p_i dx^i - a^i dp_i - |p| dt\right)\right] & 0 \\ 0 & \exp\left[\frac{i}{\hbar}\left(p_i dx^i + a^i dp_i - (-|p|dt)\right)\right] \end{pmatrix}$$

as desired.

When $A_\mu \neq 0$, let's assume $\frac{\hbar^2 E^2}{|P|^4} \approx 0$, $\frac{\hbar^2 \partial F}{|P|^3} \approx 0$.

After careful evaluation, and finally shifting $P_\mu - A_\mu \to P_\mu$, get:

$$G = \prod_t \int \frac{d^3p(t)\, d^3x(t)}{(2\pi\hbar)^3} \begin{pmatrix} (1+\vec{B}\cdot\vec{b}) \exp\left[\frac{i}{\hbar}(P_i dx^i + A_i dx^i - a^i dp_i - (|P|(1+\vec{B}\cdot\vec{b}) + A^0)dt\right] & \cdots \\ \cdots & (1-\vec{B}\cdot\vec{b}) \exp\left[\frac{i}{\hbar}(P_i dx^i + A_i dx^i + a^i dp_i - (-|P|(1-\vec{B}\cdot\vec{b}) + A^0)dt\right] \end{pmatrix}$$

The off-diagonal hopping terms do not vanish, but they are suppressed due to $\frac{\hbar^2 F}{|P|^4} \approx 0$, $\frac{\hbar^2 \partial F}{|P|^3} \approx 0$, hence we can still ignore the hopping, as in adiabatic theorem.

(The explicit $\vec{B}$ appears because, in path integral, we cannot take $\int d^3p\, d^3x\, dp_i\, dt\, A_j(x)\, e^{-\frac{i}{\hbar} dP_i dx^i} \to 0$. Instead, we need to integrate x_i by part, and get $\int d^3p\, d^3x\, (-i\hbar\, \partial_{x^i} A_j)\, dt\, e^{-\frac{i}{\hbar} dp_i x^i}$.)

For the derivation above, we know the action in EM field is

$$S = \int \left[P_i dz^i + A_{i(z)} dz^i - a^i_{(p)} dP_i - \left(|P|(1-\vec{B}\cdot\vec{b}) + A^0_{(z)}\right) dt \right]$$

The energy has a magnetic moment contribution, with gyromagnetic ratio $g=2$.

$\frac{d^3p\, d^3x}{(2\pi\hbar)^3}(1+\vec{B}\cdot\vec{b})$ is the Liouville phase space volume. The $(1+\vec{B}\cdot\vec{b})$ is $\sqrt{\det(\omega)}$ (ω is sympletic 2-form), to be explained later.

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2. Lorentz Invariance and Equations of Motion.

We started the derivation from Weyl fermion Hamiltonian, which of course is Lorentz covariant. But the action we obtain is not manifestly Lorentz invariant. What happened?

First let's cast the free particle action in a "more relativistic" form:

$$S=\int\left[p_\mu dz^\mu - a^\mu(\vec{p})\, d\vec{p}_\mu - \frac{\lambda d\tau}{2} H\right],\quad H=p_\mu p^\mu$$

Here, λ is Lagrange multiplier to enforce $H=p_\mu p^\mu=0$.

And $\vec{p}_\mu \equiv p_\mu + n_\mu (n\cdot p)$, $\partial_{p_\mu} a^\nu - \partial_{p_\nu} a^\mu = b^{\mu\nu} = -s\hbar \dfrac{\epsilon^{\mu\nu\rho\sigma}\vec{p}_\rho n_\sigma}{|\vec{p}|^3}$.
where n_μ is lab frame.

EoM: $\quad dp_\mu = 0,\quad dz^\mu = \lambda d\tau\, p^\mu \quad \left(\text{i.e. } \dfrac{dz^i}{dz^0} = \dfrac{p^i}{p^0} = \dfrac{p^i}{|\vec{p}|}\right)$

as expected.

Define $J^{\mu\nu} \equiv z^\mu p^\nu - p^\mu z^\nu + \Sigma^{\mu\nu}$, $\Sigma^{\mu\nu} = -s\hbar \dfrac{\epsilon^{\mu\nu\rho\sigma}\vec{p}_\rho n_\sigma}{|\vec{p}|} = b^{\mu\nu}|\vec{p}|^2$

$J^{\mu\nu}$ is angular momentum, $\Sigma^{\mu\nu}$ is the spin contribution

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The action depends on the lab frame n^μ in the Berry curvature term.
However, if we also shift z, the action is invariant:

$$n^\mu \to n^\mu + \beta^\mu \;\; (\beta\cdot n=0),\quad z^\mu \to z^\mu + (p\cdot n)\, b^{\mu\nu}\beta_\nu,\quad p_\mu \to p_\mu,\quad \lambda\to\lambda.$$

This means the "position" depends on choice of lab frame. (?)

e.g. $n^\mu = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$, $p^\mu = |\vec{p}|\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$, $\beta^\mu = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$, then $\delta z^\mu = \dfrac{s\hbar}{|\vec{p}|}\begin{pmatrix}0\\-1\\0\\0\end{pmatrix}$.

However, p_μ is frame ~~de~~ independent.

$J^{\mu\nu}$ is also frame independent, but the separation between "orbital contribution" and "spin contribution" is frame dependent, such that $\Sigma^{\mu\nu} n_\nu = 0$ is always satisfied.

When EM field is present,

$$S=\int\left[p_\mu dz^\mu + A_\mu(z) dz^\mu - a^\mu(\vec{p})\, dp_\mu - \frac{\lambda d\tau}{2} H\right],\quad H = p_\mu p^\mu - F_{\mu\nu}\Sigma^{\mu\nu}$$

which corresponds to $p^0 = |\vec{p}|(1-\vec{B}\cdot\vec{b})$.

Under change of frame, the action is invariant to $\mathcal{O}(\hbar)$ under:

$n^\mu \to n^\mu + \beta^\mu$, $z^\mu \to z^\mu + (p\cdot n) b^{\mu\nu}\beta_\nu$, $p_\mu \to p_\mu + F_{\mu\nu}\delta z^\nu$, $\lambda \to \lambda(1 - 2n^\rho F_{\rho\mu} b^{\mu\nu}\beta_\nu)$

$(\beta\cdot n = 0)$ $\qquad$ $(\lambda H \to \lambda H)$

Now, there is NO good notion of momentum or angular momentum.

EoM: $$\underbrace{\left(\begin{array}{c|c} F_{\mu\nu} & -\delta_\mu{}^\nu \\ \hline \delta^\mu{}_\nu & -b^{\mu\nu} \end{array}\right)}\begin{pmatrix} dz^\nu \\ dp_\nu \end{pmatrix} = \begin{pmatrix} \partial_{z^\mu} H \\ \partial_{p_\mu} H \end{pmatrix} \frac{\lambda d\tau}{2}$$

symletic 2-form ω, where ω^{-1} are Poisson Bracket elements.

$\sqrt{\det \omega} = 1 + \vec{B}\cdot\vec{b}$, gives the Liouville phase space volume.

Explicitly, in the frame $n^\mu = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, we have:

$$\begin{cases} \dfrac{dz^i}{dz^0} = \dfrac{p^i}{p^0} + \epsilon^{ijk} E_j b_k \longrightarrow \text{"anomalous velocity"} \\[2ex] \dfrac{dp^i}{dz^0} = \epsilon^{ijk} B_k \dfrac{p_j}{p^0} - E^i(1 - \vec{B}\cdot\vec{b}) + \dfrac{\partial B^k}{\partial x^i} b_k |\vec{p}| + b^i \vec{E}\cdot\vec{B} \\[2ex] p^0 = |\vec{p}|(1 - \vec{B}\cdot\vec{b}) \end{cases}$$

Chiral anomaly, where $\vec{p}=0$ acts as a monopole of Liouville flow.

Valid to $\mathcal{O}(\hbar)$. Thus, chiral anomaly is built-in as desired.

Can generalize to non-abelian gauge field, and to curved spacetime with torsion. But requires more formalism.

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3. Current.

To understand the frame dependence of position and momentum, we should ask:

What is the appropriate physical observable?
(which should be frame independent)

To observe the particle, we use the external EM field to probe it. This suggests us to look at the current:

$$j^\mu(x) \equiv \left[\frac{\delta S}{\delta A_\mu(x)}\right]_{\text{EoM}}.$$

One can show this is frame-independent to order $\hbar$, (ignoring end points of single particle worldline; end points corresponds to collisions, to be discussed later.)

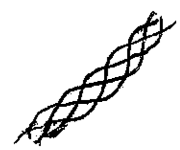
Explicitly, $\cancel{j^{\mu}_{(x)} = \int\left([dz^{\mu}]_{EOM} - \Sigma^{\mu\nu}\partial_{z^\nu}\right)\delta^4(z-x)}$

$$j^{\mu}(x) = \int\left([dz^{\mu}]_{EoM} - \lambda d\tau\, \Sigma^{\mu\nu}\partial_{z^\nu}\right)\delta^4(z-x)$$

The second term, magnetization current, is due to the magnetic moment coupling to $\vec{B}$.

If we picture $j^{\mu}_{(x)}$ by letting the δ-function have some width then, we see $j^{\mu}_{(x)}$ traces out a helical trajectory in spacetime:

trajectory of spinless particle



trajectory of spinning particle.

One can see $z^{\mu}(\tau)$ is the "central line" of the helix in spacetime. Although the helix j^{μ} is frame independent, the notion of "center" is frame dependent.

To formulate kinetic theory (collisionless), need ensemble f:
Suppose $f'(z', p') = f(z,p)$ under change of frame.
Then in

$$dN = \frac{d^4z\, d^4p}{(2\pi\hbar)^3}\sqrt{\det\omega}\cdot\frac{\delta(H)\,\theta(-p\cdot n)}{\lambda d\tau/2}\cdot f(z,p)$$

each of the three factors is frame independent.
So we can construct $J^{\mu}_{(x)} = \int dN\, dj^{\mu}_{(x)}$

$$\Rightarrow \begin{cases} J^{0}_{(x)} = \int\frac{d^3\vec{p}}{(2\pi\hbar)^3}(1+\vec{B}\cdot\vec{b})f(x,\vec{p}) \\ J^{i}_{(x)} = \int\frac{d^3\vec{p}}{(2\pi\hbar)^3}\left((1+2\vec{B}\cdot\vec{b})\frac{p^i}{|p|} + \epsilon^{ijk}E_j b_k + \epsilon^{ijk} b_k\partial_{x^j}\right)f(x,\vec{p})\end{cases}$$

anomalous density

anomalous Hall current

magnetization current

By coupling to spacetime curvature, can derive $T^{\mu\nu}$, which satisfies Ward identity $\nabla_\nu T^{\mu\nu} = F^{\mu}{}_{\nu} J^{\nu}$.

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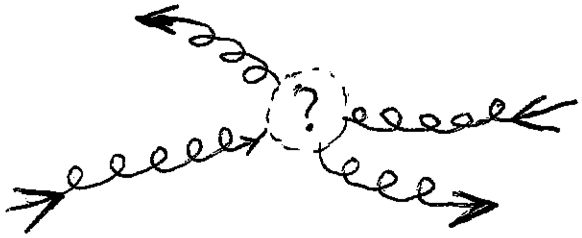
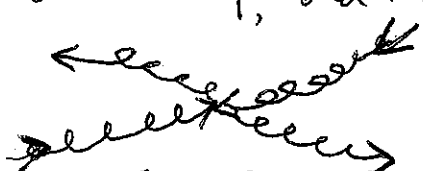
4. Collisions and H-Theorem.

Collisions are very important for application to real systems, and to discuss the relaxation to hydrodynamics. But Lorentz invariance in collisions is very non-trivial.



We already showed the current is a spacetime helix. But how to describe the end point of helix? This is highly restricted by Lorentz invariance.

The strategy is, we suppose in the CoM frame, the helical currents of the 2 in and 2 out particles just sum up, and the four $z(\tau)$'s go through a common point of collision.



CoM frame

Then, we boost the above object in spacetime into general frame.

We find an extra contribution to current, localized near the collision point, when the frame is not CoM.

Then sum up all possible collision pictures, weighted by the collision probability.

We find: (for ~~free~~ particle in no external EM field)

$$J^{\mu}(x) = \int \frac{d^3\vec{p}}{(2\pi\hbar)^3 |\vec{p}|} \rho^{\mu}(x,p)$$

$$\rho^{\mu}(x,p) = p^{\mu} f(x,p) + \Sigma^{\mu\nu} \partial_{x\nu} f(x,p) + \int_{BCD} (W_{CD\to AB} - W_{AB\to CD}) \Delta^{\mu}$$

where Δ^{μ} is the δz^{μ} needed to go from CoM frame to lab frame.

$T^{\mu\nu}$ also receive similar contribution from collisions.

In fact, $f(x,p)$ really should be interpreted as $\rho^{0}(x,p)/p^{0}$, because ρ^{μ} is the actual frame-independent object.

The effect of collision is encoded in Boltzmann's Eq.

For spinless case, $\quad p^{\mu}\partial_{\mu} f = \int_{BCD} (W_{CD\to AB} - W_{AB\to CD})$

$$W_{AB\to CD} = \frac{1}{2} |M|^2 (2\pi)^4 \delta^4(p_A+p_B-p_C-p_D)\, f_A f_B (1-f_C)(1-f_D)$$

For spinning case, f is frame dependent, so we should use that viewed in the CoM frame of collision:

$$\bar{f} = \frac{\rho \cdot n_{CoM}}{p \cdot n_{CoM}} \qquad \text{replace } f \text{ in } W \text{ by } \bar{f}.$$

$p^{\mu}\partial_{\mu} f$ should become $\partial_{\mu}\rho^{\mu}$.

The entropy current also receive contribution from collisions:

$$H^{\mu}(x) = \int \frac{d^3\vec{p}}{(2\pi\hbar)^3 |\vec{p}|} \mathcal{H}^{\mu}(x,p)$$

$$\mathcal{H}^{\mu}(x,p) = p^{\mu} \mathcal{H}(x,p) + \Sigma^{\mu\nu} \partial_{x\nu} \mathcal{H}(x,p) + \int_{BCD} (W_{CD\to AB} - W_{AB\to CD}) \Delta^{\mu} \frac{d\mathcal{H}}{df}$$

where $\mathcal{H}[f] = -f\ln f - (1-f)\ln(1-f)$.

It satisfies Boltzmann's H-theorem $\partial_{\mu}\mathcal{H}^{\mu} \geq 0$,

so the system does relax towards equilibrium

$\longrightarrow$ can derive hydrodynamics.