QCD Critical Point

1. The phase diagram and critical point
   A qualitative physical argument
   A random matrix model

2. Experimental Signatures of the Critical Point
   A brief description of the heavy ion collision
   Non-monotonicity in event-by-event fluctuation

QCD Phase Diagram

Argument:
1. Let \( \mu \rightarrow \mu_{max} \), \( m_s \) large enough
   At \( \mu=0 \), \( T \) increases,
   the hadron-QGP transition
   can be either 1st or 2nd order
   (Pisarski-Wilczek)
   Lattice supports 2nd order.
2. At \( T=0 \), increase \( \mu \), several models
   supports 1st order transition.
   But lattice doesn't work well with \( \mu \),
   so this is less affirmative.

\( \Rightarrow \) It suggests the 1st order line (from low \( T \), finite \( \mu \))
should end somewhere and becomes a 2nd order line (low \( \mu \), finite \( T \)).

Illustrate this by a random matrix model:

Eucl. time, \( Z = \int D\alpha D\phi D\chi \exp \left[ -\int d^4x \sum_{f} \tilde{T}_f (D\phi^f)\phi^f - \int d^4x (\sum_{f} F_{\alpha}^{\chi_f} - \theta F_{\alpha} F_{\chi_f}) \right] \)

In chiral basis,
In chiral basis, \( D + m_f = \begin{pmatrix} m_f & iW \\ iW^+ & m_f \end{pmatrix} \) where \( W \) is the Dirac operator on right chirality. (Regularize in \( UV \), \( W \) is an \( N_c \times N_c \) matrix; let the number of zero modes be \( N_{RL} \), then \( N_{R} - N_{L} = N_{R} - N_{L} \) is instanton number.)

The random matrix model is to replace \( \int DA \exp\left(-\frac{i}{4} \langle F_{\mu\nu} \rangle^2 \right) \) by the Gaussian weight random matrix integral:
\[
\int DW \exp\left(-\frac{N}{2} \Sigma^2 \text{Tr} \, WW^\dagger \right) \text{ where } W \text{ is that in } D + m_f, \, N = N_R + N_L.
\]
\( \Sigma \) is some Gaussian weight (will be related to \( \langle \bar{\psi} \psi \rangle \) later).

\[ Z = \int D\bar{\psi} \bar{\psi} Df \exp\left(-\frac{N}{2} \Sigma^2 \text{Tr} \, WW^\dagger \right) \exp \left(-\int dx \, \sum \bar{\psi}_f (D + m_f) \psi_f \right) \]

(I took instanton number = 0; it does not matter much in our talk.)

Perform \( DW \rightarrow \) get \( \bar{f}^m \bar{R}_f \bar{f}^n \psi^n \psi_f \) eff. int. \( (m, n \text{ are modes}) \)

Introduce \( 2N \)-quark field \( \sigma \) (Hubbard - Stöcker)

\[
\sigma_{f^+} = \sigma_{f^+} + i \sigma_{f^+}, \quad \sigma_{f^+} \leftrightarrow \bar{f}^m \bar{R}_f \bar{f}^n \psi^n \psi_f \text{ (bosonic)}
\]

Also view \( \sigma \) as an \( N_f \times N_f \) random matrix.

Perform \( Df \bar{D} \bar{f} \), get:
\[
Z = \int D\sigma \exp\left(-\frac{N}{2} \Sigma^2 \text{Tr} (\sigma \sigma^+) \right) \det^N (\sigma + m) \det^N (\sigma^+ + m)
\]

We have not included the effects of \( \mu \) and \( T \) yet.

To do so, we make a highly simplified model:

- The major effect of \( \mu \) is the non-Hermiticity of the Lagrangian.
- The major effect of \( T \) is the lowest Matsubara freq. \( \pm \pi T \).

So:
\[
D + m = \begin{pmatrix} m & iW \\ iW^+ & m \end{pmatrix} \rightarrow \begin{pmatrix} m & iW + iC \\ iW^+ + iC & m \end{pmatrix}
\]

\[ C = \begin{pmatrix} \frac{N}{4} & \frac{T + i\mu}{-T + i\mu} & \frac{0}{0} \\ \frac{T + i\mu}{-T + i\mu} & 0 & \frac{-T + i\mu}{0} \\ \frac{0}{0} & \frac{-T + i\mu}{0} & \frac{N}{4} \end{pmatrix} \]

up to normalization of \( T \) & \( \mu \).
Repeat the DW, DF, D4 integral, we get:

\[ Z = \int D\sigma \exp\left( -\frac{N}{2} \sum \text{Tr}(\sigma \sigma^T) \right) \det^{\frac{N}{4}} \left( \sigma + i T \sigma^T \right) \det^{\frac{N}{4}} \left( \sigma - i T \sigma^T \right) \]

\[ = \int D\sigma \exp\left( -\frac{N}{2} \sum \sigma \sigma^T \right) \]

\[ = \frac{1}{Z} \exp \left( -\frac{N}{2} \left( \sum \text{Tr}(\sigma \sigma^T) - (\sigma^T \sigma)^2 \right) \right) \]

Absorb \( \sum \) into \( \sigma, T, \mu \). At \( N \to \infty \) (as \( V_4 \to \infty \)), one expects the saddle point to be \( \sigma = \sigma_0 \sigma_0^T \text{I}_{N_f \times N_f} \) for real number \( \sigma_0 \).

One can see \( K \Sigma F \Sigma = \frac{1}{N_f V_4} \frac{2N}{\Delta} \Sigma \text{Tr} (\sigma \sigma^T) = \frac{N}{N_f V_4} \frac{2}{\Delta} \Sigma \text{Tr} (\sigma_0 \sigma_0^T) \)

Now find the saddle point of \( \sigma_0 \):

\[ \frac{\partial K \Sigma F \Sigma}{\partial \sigma_0} = 0 \Rightarrow \sigma_0 \left( \sigma_0^4 - 2(\mu^2 - T^2 + \frac{1}{2})\sigma_0^2 + (\mu^2 - T^2)^2 \mu^2 - T^2 \right) = 0 \]

\[ \Rightarrow \text{Gives the sssB pattern in Fig 1.} \]

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The Fireball in Heavy Ion Collision.

![Graph showing the fireball in heavy ion collision]

- After formation, the fireball will first achieve thermal equilibrium.
- It cools down via a path of nearly constant \( N_f \). It also expands (adiabatic expansion).
- When expansion rate > int. rate, freeze-out.
- Decay into lots of pions to be detected.

Notice:
- Increasing \( \sqrt{s_{\text{NN}}} \) will produce more \( S \) (entropy), since \( N \) is fixed initially, higher \( \sqrt{s_{\text{NN}}} \) makes formation at lower \( N_f \)'s, freeze-out at lower \( \mu \).
- The zig-zag on the 1st order line is due to the discontinuity of \( N_b \).
This leads to "focusing": starting with a relatively wide range of $\sqrt{s}m$, freeze-out will focus near crit. pt. with a relatively low range.

What phenomenon to observe?

Near crit. pt., $\Omega(u) \sim u^6$, has $\xi \to \infty$ (correlation length) (ideal)

$\xi$ is smaller away from crit. pt.

Thus, if some observation is monotonic in $\sqrt{s}m$ in $\xi$, then it would be non-monotonic in $\sqrt{s}m$.

Consider thermal fluctuation in an ensemble. The ensemble here is the many events of collisions. Fluctuation is appearing in event-by-event statistics. E.g., consider to pion # fluctuation.

For free pion gas, the ensemble (event-by-event) variance of $N_\pi$ is:

$\langle (\Delta N)^2 \rangle = T^2 \left. \frac{\partial \ln Z}{\partial u^m} \right|_T = T \left. \frac{\partial N}{\partial u^m} \right|_T$.

Since the momentum modes are independent,

$\langle (\Delta p^m)^2 \rangle = T \left. \frac{\partial p^m}{\partial u^m} \right|_T = \langle p^m \rangle (1 + \langle p^m \rangle)$ from B-E distribution.

$\epsilon = v_p^2$

$\langle \Delta p^m \rangle = v_p^2 \delta p^m$
This suggests that we can write an eff. prob dist. for free pion: 

\[ dP(\pi) = \prod_i d\eta_i \exp \left( -\frac{(\Delta \eta_i)^2}{2 \nu_i^2} \right) \] (for thermal fluctuations)

However, pion is not free boson, it is the Goldstone

\[ \langle \pi \rangle \sim \text{non-linear sigma} \quad (\sigma, \pi) \]

We shall expand around the minimum of \( \sigma \), which is non-zero since the freeze-out point is in the hadron phase.

\[ dP(\sigma) = d\sigma \exp \left( -\frac{m_0^2}{2T} \sigma^2 - \frac{\lambda_2}{3!T} \sigma^3 - \frac{\lambda_4}{4!T} \sigma^4 - \ldots \right) \]

\[ dP(\sigma, \pi \text{ coupling}) = d\sigma \left( -\frac{G}{T} \sigma (\pi^2 - \langle \pi^2 \rangle) + \ldots \right) \]

We ignored spatial fluctuation of \( \sigma \). The \( \pi^2 - \langle \pi^2 \rangle \) in \( \sigma, \pi \) coupling is because \( \langle \pi^2 \rangle = \sum_i \frac{1}{2} \epsilon_i \langle \eta_i \rangle \neq 0 \), and we do not want to affect the minimum of \( \sigma \).

Note that \( \pi^2 - \langle \pi^2 \rangle = \sum_i \frac{\Delta \eta_i}{\nu_i} \).

Piece up everything, we get the distribution for thermal fluctuation is:

\[ dP(\sigma, \eta_i) = d\sigma (\prod d\eta_i) \exp \left( -\sum_i \frac{(\Delta \eta_i)^2}{2 \nu_i^2} - \frac{G}{T} \sum_i \frac{\Delta \eta_i}{\nu_i} \right) \]

\[ \left( -\frac{m_0^2}{2T} \sigma^2 - \frac{\lambda_2}{3!T} \sigma^3 - \frac{\lambda_4}{4!T} \sigma^4 - \ldots \right) \]

Since this is effective, all we need is tree-level diagrams:

\[ \langle \pi_1, \pi_2 \rangle \sim \frac{m_0^2}{2T} \sigma^2 \]

\[ \langle \pi_1, \pi_2, \pi_3 \rangle \sim \frac{\lambda_2}{3!T} \sigma^3 \]

\[ \langle \pi_1, \pi_2, \pi_3, \pi_4 \rangle \sim \frac{\lambda_4}{4!T} \sigma^4 \]

Two point \( \sim \frac{m_0^2}{2T} \sigma^2 \)

Three point \( \sim \frac{\lambda_2}{3!T} \sigma^3 \)

Four point \( \sim \left( \frac{\lambda_2^2}{2} - \lambda_4 \right) \frac{m_0^2}{3!T} \sigma^4 \)

\( \left( \frac{m_0}{3} = \frac{1}{m_0} \right) \)
But this are renormalized coupling near crit. pt.
\[ \rightarrow \text{The couplings can be extracted from the universal class of } \delta_{\text{1-loop}} \]
\[ \rightarrow \lambda_3 = \lambda_3 \left( \frac{T}{T_0} \right)^{3/2}, \quad \lambda_4 = \lambda_4 \left( \frac{T}{T_0} \right)^4. \]
Thus: 3-point \( \sim 3^{9/2} \), 4-point \( \sim 3^7 \).

Does \( 3 \rightarrow \infty \) near critical point? In an infinite static system, it should. But the fireball exists in a finite time and finite size. The time for \( 3 \rightarrow \infty \) is also \( \rightarrow \infty \), which is a prediction from some universal class of dynamical crit. pheno.

It has been estimated that in the fireball, \( \xi_{\text{crit}} \sim 2 \text{ fm}, \quad \xi_{\text{big}} \sim 1 \text{ fm}. \)

So 2-point is not sensitive enough.

On the other hand, though 4-point and higher-points are sensitive,
\[ 4\text{-point } \sim \langle (\Delta N)^4 \rangle - (\langle (\Delta N)^2 \rangle)^2, \quad \text{where such subtraction} \]
\[ 0 \sim O(N) - O(N) = O(1), \quad \text{which is hard to extract from experimental noise}. \]

So it seems 3-point fluctuation is better for signal.