Descent Equations written by gauge transformations.

Weiss-Zumino consistency condition written by gauge transformations.

The BRST language

Descent Equations in BRST

solution

Geometrical meaning of BRST

WZ in BRST

Ann. of Phys. 161 (1985) 423 (main)

Weinberg, QFT II, section 22.6 (main)

Harvey, TASI 2003 Lectures, Lecture 3 [hep-th/0509097] (main)

Nakahara, Geo. Top & Phys, section 13.4, 13.5

Descent Eq: \( \text{tr}(F^{n+1}) \) satisfies \( d \text{tr}(F^{n+1}) = 0 \) (we can work in \( (2n+2) \)-dim space-time, but we don't have to.)

\( \Rightarrow \) locally \( \exists \Omega_{2n+1} \) s.t. \( d\Omega_{2n+1} = \text{tr}(F^{n+1}) \)

\( \Omega_{2n+1} \) called Chern-Simons \( (2n+1) \)-form.

Also, the gauge transff \( S_A F = [F, A] \Rightarrow S_A \text{tr}(F^{n+1}) = 0 \).

Then, consider \( S_A \Omega_{2n+1} \), \( d(S_A \Omega_{2n+1}) = 0 \Rightarrow \) locally \( \exists \Omega'_{2n+1} \) s.t. \( d\Omega'_{2n+1} = S_A \Omega_{2n+1} \).

We can draw:

\[ \Omega_{2n+1} \xrightarrow{d} \text{tr}(F^{n+1}) \xrightarrow{d} 0 \]

\[ \Omega'_{2n+1} \xrightarrow{d} S_A \Omega_{2n+1} \xrightarrow{d} 0 \]

Examples: \( n=1 \):
\( \Omega_3 = \text{Ad} A + \frac{1}{3} A^3 \), \( \Omega'_2 = \Lambda dA \)

\( n=2 \):
\( \Omega_5 = \text{Ad} A + \frac{3}{2} A^3 dA + \frac{3}{2} A^5 \), \( \Omega'_4 = \Lambda d(\text{Ad} A + \frac{A^3}{2}) \)

Will relate \( \Omega'_{2n} \) to anomalies later.

WZ consistency condition:

Consider the computation for any \( \Omega^{2n} \)

Complicated! But in fact we can just compute \( \langle \rangle \) and then use WZ consistency to get full result.
Def: \( J_0 \equiv -i \frac{\partial}{\partial x} \frac{\delta}{\delta A_{\mu}(x)} - i f_{abc} A^b(x) \frac{\delta}{\delta A_{\mu}^c(y)} = (-i \frac{\partial}{\partial x} \frac{\delta}{\delta A_{\mu}(x)}) a \)

Easy to check \([J_a(x), J_b(y)] = i f_{abc} \delta^{(x-y)} J_c(x)\).

Consider \( Z = e^{-W_{CA}} = \int [D\Phi] e^{S} \) (Euclidean space-time).

Anomaly is \( A_a(x) [A] = J_a(x) W_{CA} \).

Then: \( J_a(x) A_b(y) - J_b(y) A_a(x) = i f_{abc} \delta^{(x-y)} A_c(x) \).

Looks trivial, but useful for computation: once computed \( A \), can get full result.

In particular, in this talk we distribute anomaly equally among the current insertions, i.e. \( \frac{\delta}{\delta \phi^{(x)}} = \Gamma_{abc}^{\mu
u}(x, y, z) \), then let \( \partial_{\mu} \Gamma_{abc}^{\mu
u}(x, y, z) = \partial_{\mu} \Gamma_{bac}^{\mu
u}(x, y, z) = \partial_{\mu} \Gamma_{acb}^{\mu
u}(x, y, z) \).

One finds: \( (\partial_{\mu} J_\mu) = \frac{i}{24\pi^2} \epsilon^{\mu
u\rho\sigma} \partial_{\mu} A_{\nu} \partial_{\rho} A_{\sigma} tr(t_a t_b t_c) \).

Applying \( W_{CA} \), get:
\[
(\partial_{\mu} J_\mu) = \frac{1}{24\pi^2} \epsilon^{\mu
u\rho\sigma} tr(t_a \partial_{\mu} A_{\nu} \partial_{\rho} A_{\sigma}) \epsilon^{\tau\sigma\mu\nu} \partial_{\tau} A_{\sigma} A_{\mu} \partial_{\nu} A_{\rho} \partial_{\rho} A_{\nu} \partial_{\mu} A_{\nu} \partial_{\rho} A_{\sigma} tr(t_a t_b t_c) \]

where \( A_{\mu} = -i A^a_{\mu} t_a \).

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**BRST language:** Descend eq. and \( W_{CA} \) condition are simple in BRST language, and it shows descend eq provides candidates that satisfies \( W_{CA} \) condition (when anomaly distributed equally).

1-forms \( dx^\mu \), exterior derivative \( d \), BRST qh. \( w \), BRST transf \( S \) all anti-commutes. Notice that \( dS = -w d \).

Demand \( SA = dw + [A, \omega] = Dw \), \( Sw + \omega^2 = 0 \). Clearly \( S F = [F, w] \), \( S^2 = 0 \).

**Descent eq:**

\[
\begin{array}{c}
\Omega_{2n+1} d \rightarrow tr(F^{n+1}) d \rightarrow 0 \\
\downarrow S \downarrow S \\
\Omega_{2n+1} d \rightarrow S \Omega_{2n+1} d \rightarrow V^{-1} \\
\downarrow S \downarrow S \\
\Omega_{2n+1} d \rightarrow S \Omega_{2n+1} d \rightarrow 0 \\
\end{array}
\]

\[2/4\]
W2 condition: Define $\mathcal{A}[\omega, A] = -\frac{i}{\hbar} \int d^m x \, \omega(x) \, \mathcal{A}_0[A](x)$. Then: $\mathcal{A} = \phi W$ as functional. $W2$ consistency is $\delta \mathcal{A} = 0$ (can be seen by contracting the $W2$ condition with $-\frac{i}{\hbar} \int d^m x \, \omega(x)$).

Is this trivial, that $\delta \mathcal{A} = \phi W = 0$? This is non-trivial because $W$ is not global, so $\phi W$ should not be exact. (If exact, the $\mathcal{A}$ can be removed to some local term in action.)

So, want to find closed but not exact $\mathcal{A}$ under BRST $\delta$.

Moreover, $\mathcal{A}$ should have gh. $\# = 1$.

Try $\mathcal{A} = \int d^m x \, \delta_2 \eta$. $\text{gh} \# = 1$ ✓

$\delta \mathcal{A} = \int d^m x \, \delta \delta_2 \eta = \int d^m x \, d \delta_2 \eta_{n-1} = 0$ ✓

$\mathcal{A} \neq \delta(-\cdots)$ since $\delta_2 \eta_{n-1} \neq \delta(-\cdots)$ ✓

Thus we find a solution to the $W2$ consistency conditions.

If we want anomaly to distribute unequally, can add local terms to action to modify $\mathcal{A}$ (so that $\delta \mathcal{A}^b \delta \mathcal{A}^c$ have different coefficients for different $b, c$'s).

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**Geometrical meaning of BRST**

Starting with a 2n-dim space-time $M$ (only consider locally, so $\sim \mathbb{R}^{2n}$)

Extend a few extra dimensions, locally $\sim \mathbb{R}^p$.

Consider parameterize by $(x^\mu, \theta^\alpha), \, \mu = 1 \cdots 2n, \, \alpha = 1 \cdots p$.

Want to consider a $p$-parameter family of changes:

$A_\mu(x) \rightarrow \tilde{A}_\mu(x, \theta)$

This may or may not be gauge transf, depends on what one wants to study.

The exterior derivative on the total space: $\Delta = d + \delta$

$d \leftrightarrow \delta_\mu, \, \delta \leftrightarrow \delta_\alpha$. 
For our purpose of studying the descent eq, we can choose \( \bar{A}_n(x, \theta) \) as gauge trans. of \( A_n(x) \)

i.e. \( \bar{A}(x, \theta) = g^{-1}(x, \theta) (A(x) + d) g(x, \theta) \).

However, on total space, its natural to consider \( \Delta \).

Construct \( \bar{A}(x, \theta) = g^{-1}(x, \theta) (A(x) + \Delta) g(x, \theta) = \bar{A}(x, \theta) + w(x, \theta) \)

where \( w(x, \theta) = (g^{-1} \delta g)(x, \theta) \). Recall \( \delta \sim \delta a \), so \( w_{\alpha} = 0 \).

\( w_{\alpha} \) is a pure gauge, i.e. \( \delta w + w^2 = 0 \), as was constructed.

It is easy to check \( \mathcal{F} = \Delta \bar{A} + \bar{A}^2 = \bar{F} = d\bar{A} + \bar{A}^2 = g^{-1} F g \).

So \( \bar{F} \mu \nu = \bar{F} \mu \nu \), \( \bar{F} \mu \nu = 0 \), \( \bar{F} \alpha \beta = 0 \).

The descent eq can also be understood in this geometrical picture.

Since \( \mu, \nu = 1, \ldots, 2n \), clearly \( \text{tr}(F^{n+1}) = \text{tr}(F^{n+1}) = 0 \).

Whilst \( \text{tr}(F^{n+1}) = \delta \mathcal{K}_{2n+1}, \text{tr}(F^{n+1}) = d \Omega_{2n+1} \).

Here \( \mathcal{K}_{2n+1} \) is the CS \((2n+1)\)-form \( \mathcal{K}_{2n+1}(\bar{A}) = \mathcal{K}_{2n+1}(\bar{A} + w) \).

Expanding \( \mathcal{K}_{2n+1} \) in powers of \( w \) (i.e. counting the number of \( w \) indices out of its \((2n+1)\) indices) we have

\( \mathcal{K}_{2n+1} = \Omega_{2n+1} + \Omega_{2n+1}^1 + \Omega_{2n+1}^2 + \ldots \)

Hence \( \text{tr}(F^{n+1}) = \text{tr}(F^{n+1}) = 0 \)

\( \Rightarrow (d + \delta)(\Omega_{2n+1} + \Omega_{2n+1}^1 + \Omega_{2n+1}^2 + \ldots) = d \Omega_{2n+1} = 0 \)

Matching the terms with the same \( \delta \), we get

\[ \delta \Omega_{2n+1}^1 = d \Omega_{2n+1}^2 \]
\[ \delta \Omega_{2n+2}^1 = d \Omega_{2n+2}^3 \]

\[ \vdots \]

which are the descent eq.

Final comment: If we allow \( \bar{A}(x, \theta) \) NOT be a gauge trans. of \( A(x) \), we can study a lot more, like anomaly inflow, the topological aspect of anomaly, etc.