Descent Equations

11/7/2014 by Jingyuan Chen

Descent Equations written by gauge transformations $\longrightarrow$ Descent Equations in BRST

The BRST language

Wess-Zumino consistency condition written by gauge transformations $\longrightarrow$ WZ in BRST

Descent Equations in BRST $\xrightarrow{\text{solution}}$ WZ in BRST

Geometrical meaning of BRST $\longrightarrow$ Descent Equations in BRST

Refs: Alvarez-Gaumé & Ginsparg, Nucl. Phys. B 243 (1984) ~~449~~ 449
Ann. of Phys. 161 (1985) 423 ← (main)

Weinberg, QFT II, section 22.6 (main)

Harvey, TASI 2003 Lectures, Lecture 3 [hep-th/0509097] (main)

Nakahara, Geo, Topo & Phys, section 13.4, 13.5

Descent Eq: $\mathrm{tr}(F^{n+1})$ satisfies $d\,\mathrm{tr}(F^{n+1})=0$ (we can work in $(2n+2)$-dim space-time, but we don't have to.)

$\Rightarrow$ locally $\exists\ \Omega_{2n+1}$ s.t. $d\Omega_{2n+1} = \mathrm{tr}(F^{n+1})$

Ω_{2n+1} called Chern-Simon $(2n+1)$ form.

Also, the gauge transf $\delta_\Lambda F = [F, \Lambda] \Rightarrow \delta_\Lambda \mathrm{tr}(F^{n+1}) = 0$.

Then, consider $\delta_\Lambda \Omega_{2n+1}$, $d(\delta_\Lambda \Omega_{2n+1}) = 0 \Rightarrow$ locally $\exists\ \Omega^1_{2n}$ s.t. $d\Omega^1_{2n} = \delta_\Lambda \Omega_{2n+1}$.

We can draw:

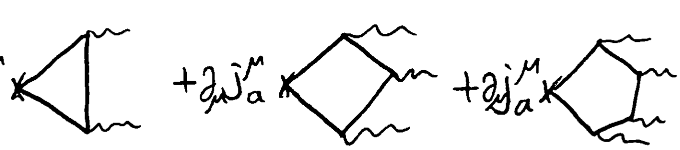
$$\begin{tikzcd}
 & \Omega_{2n+1} \arrow[r, mapsto, "d"] \arrow[d, mapsto, "\delta_\Lambda"] & \mathrm{tr}(F^{n+1}) \arrow[r, mapsto, "d"] \arrow[d, mapsto, "\delta_\Lambda"] & 0 \\
\Omega^1_{2n} \arrow[r, mapsto, "d"] & \delta_\Lambda \Omega_{2n+1} \arrow[r, mapsto, "d"] & 0 &
\end{tikzcd}$$

Examples: $n=1$: $\Omega_3 = AdA + \frac{1}{3}A^3$, $\Omega^1_2 = \Lambda\, dA$

$n=2$: $\Omega_5 = A(dA)^2 + \frac{3}{2}A^3 dA + \frac{3}{5}A^5$, $\Omega^1_4 = \Lambda\, d\left(AdA + \frac{A^3}{2}\right)$

Will relate Ω^1_{2n} to anomalies later.

WZ consistency condition:

Consider the computation for any $\partial_\mu j^\mu_a$ 

Complicated! But in fact we can just compute ◁ and then use WZ consistency to get full result.

Def: $\mathcal{J}_a(x) \equiv -i\partial_{x^\mu}\frac{\delta}{\delta A^a_\mu(x)} - i f_{abc}A^b_\mu(x)\frac{\delta}{\delta A^c_\mu(x)} = \left(-iD_{x^\mu}\frac{\delta}{\delta A_\mu(x)}\right)_a$

Easy to check $[\mathcal{J}_a(x), \mathcal{J}_b(y)] = i f_{abc}\,\delta^4(x-y)\,\mathcal{J}_c(x)$

Consider $Z_{[A]} = e^{-W[A]} = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\, e^{-S}$ (Euclidean spacetime)

Anomaly is $\mathcal{a}_a(x)[A] = \mathcal{J}_a(x)\,W[A]$.

Then: $\mathcal{J}_a(x)\,\mathcal{a}_b(y) - \mathcal{J}_b(y)\,\mathcal{a}_a(x) = i f_{abc}\,\delta^4(x-y)\,\mathcal{a}_c(x)$.

Looks trivial, but useful for computation: once computed $\triangleleft$, can get full result.

In particular, in this talk we distribute anomaly equally among the current insertions, i.e. ${}^{\mu}_{\substack{a\\(x)}}\triangleleft^{\check{b}(y)\,\nu}_{c(z)\,\rho} = \Gamma^{\mu\nu\rho}_{abc}(x,y,z)$, then let $\partial_{x^\mu}\Gamma^{\mu\nu\rho}_{abc}(x,y,z) = \partial_{x^\mu}\Gamma^{\mu\nu\rho}_{bca}(x,y,z) = \partial_{x^\mu}\Gamma^{\cdots}_{cab}$

One finds: $\left(\partial_\mu J^\mu_a\right)_\triangleleft = \frac{1}{24\pi^2}\epsilon^{\mu\nu\rho\sigma}\,\partial_\mu A^b_\nu\,\partial_\rho A^c_\sigma\,\mathrm{tr}(t_a t_b t_c)$.

Applying WZ, get:

$$\left(\partial_\mu J^\mu_a\right) = +\frac{1}{24\pi^2}\epsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}\left(t_a\left(\partial_\mu A_\nu\partial_\rho A_\sigma + \frac{1}{2}\partial_\mu A_\nu A_\rho A_\sigma - \frac{1}{2}A_\mu\partial_\nu A_\rho A_\sigma + \frac{1}{2}A_\mu A_\nu\partial_\rho A_\sigma\right)\right)$$

where $A_\mu = -iA^a_\mu t_a$.

BRST language: Descent eq. and WZ condition are simple in BRST language, and it shows descent eq provides candidates that satisfies WZ condition (when anomaly distributed equally).

1-forms dx^μ, exterior derivative d, BRST gh. ω, BRST transf δ all anticommutes. Notice that $d\delta = -\delta d$.

Demand $\delta A = d\omega + [A,\omega] = D\omega$, $\delta\omega + \omega^2 = 0$. Clearly $\delta F = [F,\omega]$, $\delta^2 = 0$.

Descent eq:

$$\begin{tikzcd}
 & & & \Omega_{2n+1} \arrow[r, mapsto, "d"] \arrow[dd, "\delta"] & \mathrm{tr}(F^{n+1}) \arrow[r, mapsto, "d"] \arrow[d, "\delta"] & 0 \\
 & & & & 0 \arrow[d, "-1"] & \\
 & & \Omega^1_{2n} \arrow[r, mapsto, "d"] \arrow[dd, "\delta"] & \delta\Omega_{2n+1} \arrow[r, mapsto, "d"] \arrow[d, "\delta"] & 0 & \\
 & & & 0 \arrow[d, "-1"] & & \\
 & \Omega^2_{2n-1} \arrow[r, mapsto, "d"] \arrow[d] & \delta\Omega^1_{2n} \arrow[r, mapsto, "d"] \arrow[d, "\delta"] & 0 & & \\
 \cdots & {} & {} & & &
\end{tikzcd}$$

WZ condition: Define $\mathcal{A}[w,A] \equiv -it_a \int d^{2n}x\, w(x)\, \mathcal{A}_a[A](x)$.

Then: $\mathcal{A} = \delta W$ as functional.

WZ consistency is $\delta\mathcal{A} = 0$ } can be seen by contracting the WZ condition with $-it_a \int d^{2n}x\, w(x)$.

Is this trivial, that $\delta\mathcal{A} = \delta^2 W = 0$? This is non-trivial because W is not global, so δW should not be exact. (If exact, the $\mathcal{A}$ can be removed to some local term in action.)

So, want to find closed but not exact $\mathcal{A}$ under BRST δ.

Moreover, $\mathcal{A}$ should have gh. # $=1$.

Try $\mathcal{A} = \int d^{2n}x\, \Omega^1_{2n}$. gh # $=1$ ✓

$$\delta\mathcal{A} = \int d^{2n}x\, \delta\Omega^1_{2n} = \int d^{2n}x\, d\Omega^2_{2n-1} = 0 \quad ✓$$

$$\mathcal{A} \neq \delta(\cdots) \text{ since } \Omega^1_{2n} \neq \delta(\cdots) \quad ✓$$

Thus we find a solution to the WZ consistency conditions.

If we want anomaly to distribute unequally, can add local terms to action to modify $\mathcal{A}$ (so that $\partial_\mu A_\nu^b\, \partial_\rho A_\sigma^c$ have different coefficients for different b, c's).

Geometrical meaning of BRST:

Starting with a $2n$-dim spacetime $\mathcal{M}$ (only consider locally, so $\sim \mathbb{R}^{2n}$)

Extend a few extra dimensions, locally $\sim \mathbb{R}^p$.

~~Consider~~ Parametrize by: (x^μ, θ^α), $\mu = 1\cdots 2n$, $\alpha = 1 \cdots p$.

Want to consider a p-parameter family of changes:

$$A_\mu(x) \longrightarrow \mathcal{A}_\mu(x, \theta)$$

This may or may not be gauge transf, depends on what one wants to study.

The exterior derivative on the total space: $\Delta = d + \delta$

$d \leftrightarrow \partial_\mu$, $\delta \leftrightarrow \partial_\alpha$.

For our purpose of studying the descent eq, we can choose $\bar{A}_\mu(x,\theta)$ as gauge transf of $A_\mu(x)$

i.e. $\bar{A}(x,\theta) = g^{-1}(x,\theta)\,(A(x)+d)\,g(x,\theta)$.

However, on total space, its natural to consider Δ.

Construct $\mathcal{A}(x,\theta) \equiv g^{-1}(x,\theta)\,(A(x)+\Delta)\,g(x,\theta) = \bar{A}(x,\theta) + \omega(x,\theta)$

where $\omega(x,\theta) = (g^{-1}\delta g)(x,\theta)$. Recall $\delta \sim \partial_\alpha$, so $\omega_\mu = 0$, ω_α is a pure gauge, i.e. $\delta\omega + \omega^2 = 0$, as was constructed.

It is easy to check $\mathcal{F} \equiv \Delta\mathcal{A} + \mathcal{A}^2 = \bar{F} = d\bar{A} + \bar{A}^2 = g^{-1}Fg$.

So $\mathcal{F}_{\mu\nu} = \bar{F}_{\mu\nu}$, $\mathcal{F}_{\mu\alpha} = 0$, $\mathcal{F}_{\alpha\beta} = 0$.

The descent eq can also be understood in this geometrical picture.

Since $\mu,\nu = 1\cdots 2n$, clearly $tr(\mathcal{F}^{n+1}) = tr(F^{n+1}) = 0$.

whilst $tr(\mathcal{F}^{n+1}) = \Delta\mathcal{X}_{2n+1}$, $tr(F^{n+1}) = d\Omega_{2n+1}$.

Here $\mathcal{X}_{2n+1}$ is the CS $(2n+1)$-form $\mathcal{X}_{2n+1}(\mathcal{A}) = \mathcal{X}_{2n+1}(\bar{A}+\omega)$.

Expanding $\mathcal{X}_{2n+1}$ in powers of ω (i.e. counting the number of α indices out of ~~the~~ its $(2n+1)$ indices) we have

$$\mathcal{X}_{2n+1} = \Omega_{2n+1} + \Omega^1_{2n} + \Omega^2_{2n-1} + \cdots$$

Hence $tr(\mathcal{F}^{n+1}) = tr(F^{n+1}) = 0$

$$\Rightarrow (d+\delta)\left(\Omega_{2n+1} + \Omega^1_{2n} + \Omega^2_{2n-1} + \cdots\right) = d\Omega_{2n+1} = 0$$

Matching the terms with the same gh #, we get

$$\delta\Omega^1_{2n} = d\Omega^2_{2n-1}$$
$$\delta\Omega^2_{2n-1} = d\Omega^3_{2n-2}$$
$$\vdots$$

which are the descent eq.

Final comment: If we allow $\bar{A}(x,\theta)$ NOT be a gauge transf of $A(x)$, we can study a lot more, like anomaly inflow, the topological aspect of anomaly, etc.