Low Energy Chern-Simons Effective Action for Abelian FQH Bulk and Edge

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1. Bulk
   \[ \nu = \frac{1}{m} \text{FQH} \]
   general FQH
   Quasi-Particle Excitation.

2. Edge
   Edge degrees of freedom
   Edge current
   Edge excitation

Main Reference: Xiao- Gang Wen [cond-mat/9506066] Chpt 2.8.3

1. Bulk

Idea: Physical phenomena know \( \rightarrow \) construct Eff. action reproducing the physical phenomena.

\[ \nu = \frac{1}{m} \text{Action} \]

In FQH, the bulk phenomenon is EM current responding to EM field as:

\[ -eJ^m = \sigma_{xy} \partial_x A^m \partial_y \]

Here \( \nu = \frac{1}{m} \), \( m = 1, 3, 5 \ldots \) for fermion (electron).

To write an eff. action reproducing such response, first we write

\[ J^m = \frac{1}{2\pi} \partial_x A^m \partial_y \]

( \( \partial_x J^m = 0 \) assumed).

Then the eff. action

\[ S = \int d^3x \left( -\frac{m}{4\pi} \sigma_{xy} \partial_x A^m \partial_y - \frac{e}{2\pi} A^m \partial_x A^m \right) \]

reproduces the \( \sigma_{xy} \) response above.

Now consider quasi-particle excitation in the \( \nu = \frac{1}{m} \) FQH liquid.

\[ \text{filling} \]

\[ \text{quasistate} \]

\[ \text{position} \]

Let the quasi-particle have an an flux equal to \( \frac{1}{m} \).

This means the following:
Label the quasi-particle current as $J^m_L$. Add the term $\int d^3x \lambda \mu J^m_L$ into the action. Then $\frac{8S}{8a_0} = 0$ gives:

$$J^0 = -\frac{e}{2\pi m} B + \frac{\ell}{m} J^m_L$$

usual piece new piece due to quasi-particle.

That's what we mean by the quasi-particle has $\gamma_m$ flux $\frac{\ell}{m}$.

(In Laughlin wave function, this means a factor $\pi (2\gamma - \delta)^L$.)

If we do $\int D\lambda \mu e^{iS}$, the result has terms

$$\int J^m_L (\theta^m)^n J^m_L \rightarrow \text{coef gives exchanging phase} \quad \Theta = \pi \frac{\ell^2}{m}.$$

$$\int \Gamma \rho \ell J^m_L \rightarrow \text{coef gives eff EM charge} \quad \Theta = -e \frac{\ell}{m}.$$

One can also see the $\Theta$ by the following argument:

A quasi particle of $l_1$ has $\gamma_m$ flux $\frac{l_1}{m}$; another of $l_2$ has $\gamma_m$ charge $l_2$. So winding the 2nd around the first gives a phase $2\pi \frac{l_1 l_2}{m}$. Taking $l_1 = l_2 = \ell$ and divide by 2 (since "winding phase" = $2\Theta$), we get $\Theta = \pi \frac{\ell^2}{m}$. Taking $l_2 = m$ for electron, we see that an electron winds around a quasi-particle has phase $2\pi \ell$, which agrees with the assertion above that such quasi-particle corresponds to the factor $\pi (2\gamma - \delta) \ell$ in wavefunction.

For single-valueness, $\ell \in \mathbb{Z}$.

Now consider a more generic setup, with $N$ species of quasi-particles in condensate. These quasi-particle may be electrons (on multiple layers), quasi-particle in electron condensate, quasi-quasi-particle in quasi-particle condensate, and so on.
\[ S = \int d^3x \left( -\frac{1}{4\pi} K_{ij} a_i \mu a_j \nu e^{i\lambda} - \frac{e}{2\pi} A_{\mu} a_i \nu a_j \lambda e^{i\lambda} \right) \]

I = 1, \ldots, n, \; K_{ij} = K_{ji}, \; entries \; of \; K \; and \; t \; are \; integers.

Thus characterizes FQH by \((K, t)\) up to \(SL(2, \mathbb{Z})\) basis transf.

Filling fraction \(\nu\) is defined s.t.:

\[-\nu eB = 2\pi t_i J_i.\]

The EoM is \(2\pi J_i = -e K_{ij}^t t_j B\), so \(\nu = t_i K_{ij}^t t_j.\)

Examples:

- \(t = (1), \; K = (m), \; \nu = \frac{1}{m}\)
- \(t = (0), \; K = (3 \; 1 \; 1), \; \nu = \frac{2}{5}\)
- \(t = (1), \; K = (3 \; 1 \; 0), \; \nu = \frac{2}{7}\)
- \(t = (1), \; K = (3 \; 2 \; 3), \; \nu = \frac{2}{5}\)
- \(t = (1), \; K = (3 \; 1), \; \nu = \frac{1}{2}\)

\([\text{The two } \nu = \frac{2}{5} \text{ are related by } (1, 0) \in SL(2, \mathbb{Z}). \text{ So they are the same; but if we also consider spin of quasi-particles, they have } S = (\frac{1}{2}) \text{ and } S = (\frac{\nu}{2}) \text{ which are NOT related by } (1, 0). \text{ So they are the same if spin is absent (or interaction mixes spins), and are different if spin is in consideration.}\]

General FQH

Quasi-particle

Again consider a single quasi-particle \(g\) with coupling \(\int d^3x \; l_i a_i \mu J_i^\nu\).

Then it has a \(\nu_i\) charge \(l_i\)

\[ a_i \mu \text{ flux } K_{ij}^t l_j \]

exchange phase \(Q = l_i K_{ij}^t l_j\)

\(\nu_i\) EM charge \(Q = -e t_i K_{ij}^t l_j\).
Edge Dof

\[ S = -\frac{m}{4\pi} \int d^3x \, a_\mu \partial_\nu \phi \epsilon^{\mu\nu\lambda} \quad \text{under} \quad a_\mu \rightarrow a_\mu + \partial_\mu \phi \]

\[ \Delta S = \frac{m}{4\pi} \int \text{boundary} \, d\sigma \, dt \, \hat{n}_\mu \partial_\nu \phi \epsilon^{\mu\nu\lambda} \]

**Note**

\[ \Delta S = 0 \quad \iff \quad f = \text{const. on boundary} \]

\[ \implies \partial_t f = \partial_\sigma f = 0 \quad \text{on boundary} \]

\[ \implies \partial_t \text{and} \partial_\sigma \text{do NOT transform on boundary.} \]

To see this in another way, EoM says \( \partial_\lambda \epsilon = 0 \),
i.e. \( \partial_\lambda = \partial \phi \).

In bulk, \( \phi \rightarrow \phi + f \) is gauge inv., so
\( \phi \) in bulk are NOT dof. On edge, \( \phi \) is dof (up to const shift).

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**Edge Action & Current**

What is the dynamics of edge dof?

Physical phenomenon: Edge current of velocity \( V \).

(\( V \) is non-universal, magnitude is an input)

We claim that restricting on boundary

\[ \partial_t + V a_\sigma = 0 \]

will reproduce the physical phenomenon.

To see this, now, with this restriction and bulk gauge transf.,
we can make \( \partial_t + V a_\sigma = 0 \) everywhere. Then one find

\[ S = -\frac{m}{4\pi} \int d\sigma \, dt \, (\partial_t + V a_\sigma) \phi \cdot \partial_\sigma \phi \]

as an edge eff. action

EoM: \( (\partial_t + V a_\sigma) (\partial_\sigma \phi) = 0 \).

If we can identify \( \partial_\sigma \phi \) as current density on edge,
the EoM would indeed reproduce the physical phenomenon.
To see $P = \frac{1}{2\pi} \partial \sigma \phi$ is edge current, consider
\[ \int d^3x \; \frac{e}{2\pi} \, \partial \sigma \phi \cdot A \in \mathbb{R}^N \quad (U(1) \text{ gauge inv. is explicit}) \]
Again imposing $A_t + V\phi = 0$, we get
\[ -\frac{e}{2\pi} \int d\sigma \, dt \, (A_t + V\phi) \partial \sigma \phi. \]
So $P = \partial \sigma \phi$ indeed is the edge current.

To quantize, notice that $H = \frac{mV}{4\pi} \int d\sigma \, (\partial \phi)^2 = \pi mV \int d\sigma \, \rho^2$.
Since $i[H, P] = \partial_t P$, to reproduce the EoM, we need:
\[ i [P(\sigma), P(\sigma')] = \frac{e}{2\pi} \, \partial_\sigma \delta(\sigma - \sigma') \]
\[ \Leftrightarrow \quad i [\phi(\sigma), \phi(\sigma')] = -\frac{\pi}{m} \, \text{Sign} (\sigma - \sigma') \]
(Kac–Moody Algebra). And $mV > 0$ for $H$ to be bounded below.

More generally,
\[ S = \frac{1}{4\pi} \int d\sigma \, dt \, (K_{IJ} \partial_t \phi_I \partial_\sigma \phi_J - V_{IJ} \partial_\sigma \phi_I \partial_\sigma \phi_J) \]
$V_{IJ}$ non-universal, $V_{IJ}$ positive-definite.
\[ i [P_I(\sigma), P_J(\sigma')] = \frac{k_{IJ}}{2\pi} \, \partial_\sigma \delta(\sigma - \sigma'). \]

One can find a matrix $U$ s.t.
\[ (U K U^T)_{IJ} = \delta_I \delta_J \quad (\delta_I = \pm 1) \]
\[ (U V U^T)_{IJ} = |V| \delta_I \delta_J \quad (|V| > 0) \]
Let $V_I = \sigma_I |V|$, $\sigma_I$ determines the direction of edge mode
$\sigma_I$ is universal, $|V|$ is NOT.
An argument of universality will be given later.

In bulk, $|V| \delta_I J^a_I$ gives an excitation of $\alpha_\mu$ flux $K_{IJ}^a J^a_J$.
On edge, we look for an operator $\pm 1/2$ that does this.
This means: \( i [ \Psi_1 (\sigma), \bar{\Psi}_1 (\sigma')] = \sum_j K_{j1}^1 \delta (\sigma - \sigma') \Psi_1 (\sigma') \)

\[ \Rightarrow \Psi_1 \propto e^{i \lambda} \phi_1, \quad \bar{\Psi}_1 \propto \Psi_1^\dagger. \]

Then one can work out \( \bar{\Psi}_1 (\sigma) \Psi_1 (\sigma') = (-1)^{\lambda} \bar{\Psi}_1 (\sigma') \Psi_1 (\sigma) \)

where \( \lambda \equiv \Sigma_j \lambda_j K_{j1}^1 \) is the edge statistics.

And the excitation has charge \( Q = \Sigma \lambda_j K_{j1}^1 \).

Write \( \lambda_j = U_{ij} \lambda_j \), \( e_j = U_{ij} \), \( \Rightarrow \lambda = \Sigma_i \lambda_i e_i \).

Thus provides a universal argument for the Kac-Moody Alg.

\begin{align*}
\text{Fermi} \quad & \text{bring excitation } \#2 \text{ from } \sigma_2 \text{ to } \sigma_2' \\
\text{Vac} \quad & \text{#} \quad \text{from } \sigma_2 \text{ to } \sigma_2' \\
\Rightarrow \text{effectively wind } \#1 \text{ around } \#2, \\
& \text{should get phase } 2\pi \Sigma_j \lambda_j K_{j1}^1 e_j \\
\Rightarrow \text{Express this in } \Psi_j = e^{i \lambda_j} \text{ and working backwards, get the Kac-Moody alg.} \\
& \text{since the statistics should be universal, so is the alg.}
\end{align*}

One can use the commutation relations to find the propagator \( \langle \Psi_1^\dagger (\sigma, t) \Psi_1 (0, 0) \rangle \propto (\sigma - \lambda_1 t + i \sigma_2 \xi)^{-\sum_i \lambda_i^2} \)

For \( \sigma = 0 \), \( \langle \Psi_1^\dagger (0, t) \Psi_1 (0, 0) \rangle \propto t^{-\sum_i \lambda_i^2} \equiv t^{-q_e} \)

The power is NOT \(-1\), but \(\sum_i \lambda_i^2\). This can have experimental consequences (has been observed).

For system with all \( \sigma = 1 \) \((-1) \)

\[ \Rightarrow q_e = \lambda \epsilon (-\lambda \xi), \text{ universal,} \]

Otherwise \( q \) depends on non-universal \( V_{ij} \) via \( \lambda \).