Quantum-classical mapping

We will show that

d-dimensional $\Longleftrightarrow$ (d+1)-dimensional statistical mechanics system.

For quantum Hamiltonian $H$ at temperature $T$,

$$ Z = T \sum_m e^{-\beta H} = \sum_m <m| e^{-\beta H} |m> $$

where $|m>_T$ is same basis.

Break up $e^{-\beta H} = e^{-\beta A - \beta B - \beta C}$ where $N\beta = F$.

There are $O(d^2)$ terms that we are ignoring for small $\beta$ later on.

Insert resolutions of identity to get

$$ Z = \sum_{m,m',m''} <m| e^{-\beta A}|m'> e^{-\beta B}|m''>_T e^{-\beta C}|m''>_T $$

For the transverse field quantum Ising model in 1D

$$ H = -J \sum_{x<} \hat{\sigma}^x_x \hat{\sigma}^x_{x+1} - \sum_{x} H_x $$

With the basis of $\hat{\sigma}^z_x \hat{\sigma}^z_{x+1} |m>_T = |S_x(1), S_{x+1}(2)>$.

Work in the basis of $\hat{\sigma}^z_x \hat{\sigma}^z_{x-1} |m>_T = |S_x(1), S_{x+1}(2)>$.

For a single time step,

$$ \langle m_{x+1} | e^{-\beta A} | m_x > = \langle m_{x+1} | e^{-\beta A} | m_x > $$

$$ = \delta_{x+1} e^{-\beta (H_x + H_{x+1})} \langle m_{x+1} | e^{-\beta A} | m_x > $$

$$ = \delta_{x+1} e^{-\beta (H_x + H_{x+1})} \langle S_{x+1}(1) S_{x+1}(2) | e^{-\beta A} | S_x(1), S_{x+1}(2) > $$

For each site,

$$ \langle S_x(1), S_{x+1}(2) | e^{-\beta A} | S_x(1), S_{x+1}(2) > $$

$$ = \sum_{S_{x-1}} \langle S_x(1), S_{x+1}(2) | e^{-\beta A} | S_x(1), S_{x+1}(2) > $$

One can show that $\langle S_x | S_x > = \frac{1}{S} e^{-\beta A}$.

Note that $\langle S_x | S_x >$ is real, so $\langle S_x | S_x > = \frac{1}{S} e^{-\beta A}$ as well.
\[
\begin{align*}
\langle S^z_i | e^{i E^{1/2} \delta z} | S^z_j \rangle &= \sum_{\sigma, \delta z} e^{i E^{1/2} \delta z} \frac{1}{2} e^{i \frac{1}{2} \delta z (1 - \frac{5}{3} \sigma + \frac{2}{3} \delta z)} \\
\quad = \frac{1}{2} \left( e^{i E^{1/2} \delta z} + e^{-i E^{1/2} \delta z} \right) \frac{1}{2} e^{i \frac{1}{2} \delta z (1 - \frac{5}{3} \sigma + \frac{2}{3} \delta z)} \\
\quad = \frac{1}{2} \left( e^{i \delta z} + e^{-i \delta z} S^z_i S^z_j \right) 0
\end{align*}
\]

Try to put this into the form
\[
\langle S^z_i | e^{i E^{1/2} \delta z} | S^z_j \rangle = c E^{1/2} \delta z = c (\cosh J z + S^z_i S^z_j \sinh J z) 
\]

Comparing 0 and 0 gives (divide the coefficient of \(E^{1/2}\))
\[
c = \tanh J z
\]

So, for the transverse field quantum Ising model,
\[
Z = \sum_{\{S^z_i(1,2), S^z_i(2,1), \ldots, S^z_i(L,1)\}} \cosh \left( \sum_{i=1}^{L-1} J (S^z_i(1,2) S^z_i(2,1) + \sum_{i=1}^{L-1} J^z_i S^z_i(i,2) S^z_i(1,2)) \right)
\]

For large \(\beta\), the infinite lattice

A dimensional quantum Ising \(\to\) a 1-dim classical Ising with transverse field

For small \(J\), small, \(J^z\), large, anisotropic Ising

Isotropic classical Ising

\[
I_{z=1} = I_z = K_z \Rightarrow K = \tanh^{-1} \left( e^{-2K_z} \right)
\]

\[
Z = \sum_{\{S^z_i(1,2), S^z_i(2,1), \ldots, S^z_i(L,1)\}} \exp \left[ K \left( \sum_{i=1}^{L-1} S^z_i(1,2) S^z_i(2,1) + \sum_{i=1}^{L-1} S^z_i(i,2) S^z_i(1,2) \right) \right]
\]

\(K\) is the effective inverse temp for classical model.
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<td>[ \lambda ] [ \sim ] large [ \lambda ]</td>
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E.g. 0-dim quantum Ising is just single spin \( H = -\lambda \sigma \cdot \sigma \).

1st excited \( \{-\sigma\} \) \( E = 0 \) \( \Rightarrow P = e^{\beta 0} + e^{-\beta 0} \)

No phase transition, just a gapped quantum system.

The equivalent 1d classical Ising with

\[ K = e^{-\lambda_1} (e^{-\lambda_2}) \]

is always ordered for all \( T > 0 \).
Self-duality of isotropic Ising model (classical 2D)

High Temp: $Z(k) = \sum_{\sigma \ldots \sigma_n} \exp \left[ \frac{\beta \sum \sigma_i \sigma_j}{2} \right]$ \text{ and } k = \beta J \text{ is small.}

Note that $\exp(\beta \sigma_i \sigma_j) = \cosh \beta + \sigma \sinh \beta$.

$\Rightarrow Z(k) = \sum_{\sigma_1 \ldots \sigma_n} \prod_{i < j} (\cosh \beta + \sigma_i \sigma_j \sinh \beta) \quad \sum_{\sigma_i} \sigma_{\text{odd}} = 0$

$= \left( \cosh \beta \right)^{2^n - n} \sum_{\sigma_1 \ldots \sigma_n} \prod_{i < j} (1 + \sigma_i \sigma_j \tanh \beta). \quad \sum_{\sigma_i} \sigma_{\text{even}} = 0$

$\prod_i (1 + \sigma_i \sigma_j \tanh \beta)$ is a sum of terms where each term has a 1 or $\sigma_i \sigma_j \tanh \beta$ on each site. If a term has odd number of $\sigma_i \sigma_j \tanh \beta$ on each site, it gives zero. So, need even number of $\sigma_i \sigma_j$ on each site. The non-zero terms correspond to closed loops.

$Z(\beta) (\cosh \beta)^{-2n} (\tanh \beta)^{2n} \log^{\text{th}} \text{ of loop.}$

$= 1 + N (\tanh \beta)^4 \times 2^N (\tanh \beta)^2 \times \frac{1}{3} N \left( N - 1 \right) \left( N - 2 \right)$

Low Temp: Choose $T = 0$ ground states to be all up. Expand in number of spin flips. Each spin flip corresponds to a broken bond with energy cost $\Delta J$.

$Z(\beta) e^{-\beta \Delta J} = \sum_{\text{boundary of droplets}} e^{-\Delta J \sum_{\text{boundary of droplets}}}$

Low Temp: Choose $T \to 0$ ground states to be all up. Expand in number of spin flips. Each spin flip corresponds to a broken bond with energy cost $\Delta J$. 

$Z(\beta) e^{-\beta \Delta J} = \sum_{\text{boundary of droplets}} e^{-\Delta J \sum_{\text{boundary of droplets}}}$
The series are the same if we identify \( \text{tanh} k = e^{-2k^*} \).

\[ \sum \left( k^* \right) = \sum \left( k \right) \]

Manipulating the hyperbolic functions gives:

\[ \text{sinh}(2k) = \text{sinh}(2k^*) \cdot \frac{Z(k^*)}{Z(k)} \]

\[ \text{cosh}(2k) = \text{cosh}(2k^*) \cdot \frac{Z(k^*)}{Z(k)} \]

Kramers-Wannier duality:

If the critical point is unique, then it must occur at \( k = k^* \).

(Note that we made expansions, not approximations, so the series is exact.) Then:

\[ \text{sinh}^2(k) = 1 \Rightarrow e^{2k} = 5 + 1 \]

Unfortunately, does not give critical exponents.

For finite lattice, have sum of terms, hence convergent, hence duality is exact at any \( T \).

Also works on triangular and homogenous lattices.

\begin{align*}
\text{Low } T \text{ triangle} & \quad \text{High } T \text{ hexagon} \\
\text{\includegraphics[width=0.2\textwidth]{triangle.png}} & \quad \text{\includegraphics[width=0.2\textwidth]{hexagon.png}}
\end{align*}
\[ H(\sigma) = 2H_1(\sigma) \]

Note, we solve the 1D quantum Ising exactly to show that mass gap vanishes at critical point. If critical point is given, then it vanishes at 2, \( x = 2 \). For \( n \), we have \( m_{\text{even}} \) and \( m_{\text{odd}} \) overlap. Because only even number of mass overlap, we check if adjacent spins are adjacent. Can be shown that dual operators satisfy the same Pauli algebra. For convenience, define operators on dual lattice. Let \( x = 0 \) and \( n \).
Exact solution of 1D quantum Ising (Jordan-Wigner) 

\[ H = - \sum \delta_{x}(n) - 2 \sum \delta_{x}(m) \delta_{x}(n) \]

Turn this into a free form:

Define raising, lowering:

\[ \sigma_{+}(n) = \frac{1}{2} \left[ \sigma_{x}(n) + \sigma_{y}(n) \right] \]
\[ \sigma_{-}(n) = \frac{1}{2} \left[ \sigma_{x}(n) - \sigma_{y}(n) \right] \]

Label lattice sites \( n = -N, -N+1, \ldots, N \).

\[ C(n) = \prod_{j=-N}^{N} \exp \left[ i \times \frac{\theta_{j}}{2} \sigma_{x}(j) \sigma_{y}(j) \right] \sigma_{-}(n) \]
\[ C^{-1}(n) = \sigma_{+}(n) \prod_{j=-N}^{N} \exp \left[ -i \times \frac{\theta_{j}}{2} \sigma_{x}(j) \sigma_{y}(j) \right] \]

Idea: \( \sigma_{+}, \sigma_{-} \) anticommutate on same site and square to zero. Want to form a string of operators so they anticommutate everywhere, like fermions. Observe that:

\[ \sigma_{-}(n) \sigma_{+}(n) = \frac{1}{2} \left[ \sigma_{x}(n) - \sigma_{y}(n) \right] \]
\[ \sigma_{+}(n) \sigma_{-}(n) = \frac{1}{2} \left[ \sigma_{x}(n) + \sigma_{y}(n) \right] \]
\[ \exp \left[ i \times \frac{\theta_{j}}{2} \sigma_{x}(j) \sigma_{y}(j) \right] \]

Check for \( \sigma_{z} \equiv 1 \).

Allow us to write:

\[ C(n) = \prod_{j=-N}^{N} \left[ - \delta_{x}(j) \right] \sigma_{-}(n) \]
\[ C^{-1}(n) = \sigma_{+}(n) \prod_{j=-N}^{N} \left[ - \delta_{x}(j) \right] \]

Easy to verify the usual anticommutation relations:

\[ \{ C(m), C^{+}(m) \} = \delta_{nm} \]
\[ \{ C(n), C(m) \} = 0 \]

For \( n \neq m \),

\[ C(m) C^{-1}(n) C^{-1}(m) C(n) = \sigma_{-}(n) \sigma_{+}(n) + \sigma_{+}(n) \sigma_{-}(n) = 1 \]

For \( n < m \),

\[ C(m) C^{+}(n) C^{+}(m) C(n) = \sigma_{-}(m) \sigma_{+}(m) \sigma_{-}(m) \sigma_{+}(m) = 0 \]

Write the terms at \( C, C^{-1} \). Using (14.86),

\[ \sigma_{x} \sigma_{x} + \sigma_{y} \sigma_{y} - 1 = 0 \]

Note that:

\[ C^{-1}(n) C(n-1) = \sigma_{+}(n) \left[ - \sigma_{z}(n) \right] \sigma_{+}(n-1) \]

But:

\[ \sigma_{+}(n) \sigma_{z}(n) = - \sigma_{+}(n) \]

So:

\[ C^{-1}(n-1) C(n-1) = \sigma_{+}(n) \sigma_{-}(n) \]

Can similarly show that:

\[ C(n) C^{-1}(n-1) = \sigma_{+}(n-1) \sigma_{-}(n-1) \]
\[ C^{-1}(n) C^{-1}(n+1) = \sigma_{+}(n) \sigma_{-}(n+1) \]
\[ C(n) C(n+1) = - \sigma_{+}(n) \sigma_{-}(n+1) \]

\[ C(n) C(n-1) = - \sigma_{+}(n) \sigma_{-}(n-1) \]

\[ C^{-1}(n) C^{-1}(n+1) = \sigma_{+}(n+1) \sigma_{-}(n+1) \]

\[ C^{-1}(n) C^{-1}(n-1) = \sigma_{+}(n) \sigma_{-}(n) \]

\[ C(n) C(n+1) = - \sigma_{+}(n) \sigma_{-}(n+1) \]

\[ C(n) C(n-1) = - \sigma_{+}(n) \sigma_{-}(n-1) \]
Our coupling term becomes
\[ \sigma_n(m)\sigma_{n+1}(m+1) = [\sigma^+(m) + \sigma^-(m)] [\sigma^+(m+1) + \sigma^-(m+1)] \]
\[ = [c^+(m) - c^-(m)] [c^+(m+1) + c^-(m+1)] \]
We get a quadratic fermionic Hamiltonian.
\[ H = -2 \sum_n c^+(n)c(n) - 2 \sum_n [c^+(n)c(n)] [c^+(n+1)c(n+1)] \]

Having translational \( \mathbf{v}_k \), take Fourier transform to diagonalize.
\[ c(n) = \sqrt{\frac{1}{2N+1}} \sum_k e^{-i \mathbf{k} \cdot \mathbf{a}_n} \quad k = 0, \pm \frac{2\pi}{2N+1}, \pm \frac{4\pi}{2N+1}, \ldots, \pm \frac{2\pi N}{2N+1} \]

We show that \( a^+_k \) are fermionic i.e. \( [a^+_k, a^+_l] = 0 \)
\[ a^+_k a^+_l = \delta_{kl} a^+_k \]

\[ \sum_n c^+_n c^+_{n+1} = \frac{1}{2N+1} \sum_n \sum_k e^{i \mathbf{k} \cdot \mathbf{a}_n} e^{i \mathbf{k} \cdot \mathbf{a}_{n+1}} a^+_k a^+_l \]

\[ \mathcal{L} \rightarrow \sum_k e^{-i \mathbf{k} \cdot \mathbf{a}_n} a^+_k a^+_l \]

\[ \mathcal{L} \rightarrow \sum_k e^{-i \mathbf{k} \cdot \mathbf{a}_n} a^+_k a^+_l \]

\( \mathcal{L} \) contains terms as well. \( \mathcal{L} \rightarrow \)
\[ H = -2 \sum_k a^+_k a_k - 2 \sum_k (e^{-i \mathbf{k} \cdot \mathbf{a}_n} a^+_k a_k + e^{-i \mathbf{k} \cdot \mathbf{a}_n} a^+_k a_k + e^{i \mathbf{k} \cdot \mathbf{a}_n} a_k a_k) \]
\[ = -2 \sum_k (H_2 \cos k) a_k a_k - 2 \sum_k (e^{-i \mathbf{k} \cdot \mathbf{a}_n} a^+_k a^+_k - e^{i \mathbf{k} \cdot \mathbf{a}_n} a_k a_k) \]

Transform \( a_k, a_k^+ \) into new set of fermionic operators \( n_k = U_k a_k + i V_k a_k^+ \)
\[ n_{k+1} = U_{k+1} a_{k+1} - i V_{k+1} a_{k+1}^+ \]
\[ n_k^+ = U_k a_k - i V_k a_k^+ \]
\[ n_k^+ = U_k a_k - i V_k a_k^+ \]
\[ n_{k+1}^+ = U_{k+1} a_{k+1} + i V_{k+1} a_{k+1}^+ \]
\[ n_{k+1}^+ = U_{k+1} a_{k+1} + i V_{k+1} a_{k+1}^+ \]
Then,
\[ \{n_k, n_{k'}\} = \delta_{kk'} \]
\[ \{n_k^+, n_{k'}\} = \{n_k^+, n_{k'}^+\} = 0 \rightarrow U_k^+ V_k = 1 \] \( (4.111) \)
Write \( H \) in terms of these operators
\[ H = \sum_{k>0} \[ - (H_2 \cos k)(U^2 - V^2) + 4 \text{sink} \ U_k V_k \] (\( n_k^+ n_k + n_k n_k^+ \))
\[ + \sum_{k>0} \[ 4i(H_2 \cos k)U_k V_k + 2 \text{sink} (U_k^2 - V_k^2) \] (\( n_k^+ n_{k'} + n_k n_{k'}^+ \))
To make this diagonal, i.e. only \( n^+ n \) term, need
\[ U (H_2 \cos k) U_k V_k + 2 \text{sink} (U_k^2 - V_k^2) = 0 \] \( (4.114) \)
By \( (4.111) \), write \( U_k = \cos \Theta_k \quad V_k = \text{sink} \Theta_k \)
Then, \( (4.114) \) becomes \( \cos \Theta_k (H_2 \cos k) \Theta_k + 2 \text{sink} \cos \Theta_k = 0 \)
\[ \rightarrow \tan \Theta_k = - \frac{2 \text{sink}}{1 - H_2 \cos k} \]
Can choose sign such that
\[
\sin(D\Omega_x) = \frac{2 \sin k}{\sqrt{1 - 2 \cos k - 1}}
\]
\[
\cos(D\Omega_x) = -\frac{1 + 2 \cos k}{\sqrt{1 - 2 \cos k + 1}}
\]
\[
H = \frac{2}{2} \sqrt{1 + 2 \cos k + 1}^2 \eta_k^* \eta_k + \text{const.}
\]
\[
\Lambda_k = 2 \sqrt{1 + 2 \cos k + 1}^2
\]
Minimum \( \Lambda_k \) at \( k = \frac{\pi}{4} \)
where \( \Lambda_{2\pi} = 2 \pi - 2 \) gap.
Since \( \Lambda \sim \frac{1}{\Lambda} \), and \( 2^{\frac{1}{\Lambda_2}} \),

\( \nu = 1 \).
Ising gauge theory: D dimensional lattice with spins on links.

\[ H = -J \sum_{n,m,v} \sigma_3(n,m) \sigma_3(n+m,v) \sigma_3(n+m+1,v,-m) \sigma_3(n+1,v,-v) \]

Gauge transform \( G(n) \) flips all spins on links at \( n \).

\[
\begin{array}{c}
\uparrow \\
\downarrow \\
\uparrow \\
\downarrow \\
\end{array} \quad \begin{array}{c}
G(n) \\
\downarrow \\
\uparrow \\
\downarrow \\
\end{array}
\]

Since the links at each site can border a plaquette two or zero times, we see that applying \( G \) does not affect any of the plaquettes.

\[
(\prod_3(i)) = 1 \Rightarrow 7! \sigma_3(i) = \pm 1 . \quad "-1" \ \text{is frustrated.}
\]

Elitzur's theorem: Local symmetry cannot break spontaneously.

In this context \( \langle G_2 \rangle \) vanishes identically at all \( T \), so we cannot use it as an order parameter.

Proof: To see if spontaneous magnetization is possible, apply external magnetic field, which results in \( h \sum \sigma_3(n,m,v) \).

Take \( h \to 0 \). If \( \lim_{h \to 0} \langle \sigma_3(n,m,v) \rangle \neq 0 \), then system is magnetized. If \( \langle \sigma_3(n,m,v) \rangle = 0 \), then not only have spontaneous breaking of global up\equiv down symmetry, but also the local symmetry as well.

Consider

\[
\langle \sigma_3(n,m,v) \rangle_h = \frac{\sum_{\text{config}} \exp \left[ \beta \sum \sigma_3(n,m,v) \sigma_3(n+m,v) \sigma_3(n+m+1,v,-m) \sigma_3(n+1,v,-v) \right]}{\sum_{\text{config}} \exp \left[ \beta \sum \sigma_3(n,m,v) \sigma_3(n+m,v) \sigma_3(n+m+1,v,-m) \sigma_3(n+1,v,-v) \right].}
\]

Consider a local gauge transform at site \( n \) by \( \{K_n\} \).

Plaquette terms invariant by external field term becomes

\[
h \sum \sigma_3 = h \sum \sigma_3 - h \sum \sigma_3 \]

where \( \sigma_3' \) is transformed spin.

- \( \sigma_3'(n) = \sigma_3(n) \), if \( n \in \{K_n\} \).
- \( \sigma_3'(n) = \sigma_3(n) \), if \( n \notin \{K_n\} \).
- \( \sigma_3'(n) = 0 \), if \( n \notin \{K_n\} \).
\[ \langle \sigma_2(n, \nu) \rangle_h = -\sum \langle \sigma_3'(n, \nu) \rangle \exp \frac{1}{\beta} \sum \sigma_3 \sigma_2' \sigma_3' \sigma_2' + h \sum \sigma_3 \delta \sigma_3 \sum \sigma_3 \]  

\[ = \langle -\sigma_2(n, \nu) \exp \left[ -h \sum \sigma_3 \delta \sigma_3 \right] \rangle_h \]

Now, we can get an upper bound
\[ |\langle \sigma_2(n, \nu) \rangle_h - \langle -\sigma_2(n, \nu) \rangle_h| \]
\[ = |\langle -\sigma_2(n, \nu) \left[ \exp \left[ -h \sum \sigma_3 \delta \sigma_3 \right] - 1 \right] \rangle_h| \]
\[ \leq \left[ e^{\frac{h}{\beta}} - 1 \right] \left| \langle \sigma_2(n, \nu) \rangle_h \right| \xrightarrow{h \to 0} 0 \]

\[ \Rightarrow \langle \sigma_2(n, \nu) \rangle_h \to 0 = \langle -\sigma_2(n, \nu) \rangle_h \to 0. \]

\[ \Rightarrow \langle \sigma_2(n, \nu) \rangle = 0. \]

Since we have no local order parameter, how to distinguish phases?

Wegner suggests to look at spatial dependence of correlation functions.

Inspired by 2D XY model, phase transition without spontaneous magnetization.

Low Temp: \[ \langle s(0), s(n) \rangle \propto |n|^{-k/\beta + 1} \]

High Temp: \[ \langle s(0), s(n) \rangle \propto \exp \left( -\frac{|n|}{\tau(\beta)} \right) \]

Different behavior \( \Rightarrow \) must have phase transition.

Consider a gauge-invariant correlation function
\[ \langle \prod_{\ell \in C} \sigma_3(\ell) \rangle \] where \( C \) is a closed loop.

\[ \langle \prod_{\ell \in C} \sigma_3(\ell) \rangle = \frac{\sum \prod_{\ell \in C} \left( 1 + \sigma_3 \sigma_2 \sigma_2' \sigma_3' \right) \prod_{\ell \in C} \sigma_3}{\sum \prod_{\ell \in C} \left( 1 - \sigma_3 \sigma_2 \sigma_2' \sigma_3' \right) \prod_{\ell \in C} \sigma_3} \]

\( \langle \prod_{\ell \in C} \sigma_3(\ell) \rangle \) only surviving term is \( \sum |A_1| = 2^n \) to lowest order.
Numerator: 1st non-vanishing contribution is a minimal surface bounded by $\beta$.

So, $\langle \mathcal{T}_c \sigma_3 \rangle = (\tanh \beta)^{N_c}$ where $N_c$ is # of square in minimal surface.

Also, $\exp \left\{ \frac{1}{2} \ln(\tanh \beta) \right\} A^2$ is a complicated function of $\beta$.

To higher orders, third $\langle \mathcal{T}_c \sigma_3 \rangle = \exp \frac{1}{2} - \frac{1}{4}(\beta) A^2$

Area law

Low temp and $d > 2$

For each sort of gauge equivalent configurations, we can pick a representative spin configuration. So, when summing over spin config, just sum over representative spin config and multiply by same multiplicity factor.

As: what is this multiplicity factor? Given a reference spin config $\langle \sigma_2 \rangle$, any other config that is gauge equivalent to this can be expressed as

$G^n(1) \ldots G^n(N) \langle \sigma_2 \rangle$ where $n_1, \ldots, n_N = 0,1$.

So, for all equivalence classes of spin configs, the multiplicative factor is the same i.e. $n$.

This multiplicity factor occurs in both numerator and denominator, so HT cancels out.

Represent the $\mathcal{T} = 0$ spin config as all spin up, and expand in # of flipped spins.

$\langle \mathcal{T}_c \sigma_3 \rangle = \sum \frac{1}{\mathcal{Z}} \mathcal{T}_c \sigma_3 \exp \frac{1}{2} \beta \sum \sigma_2 \sigma_3 \sigma_4 ...$

1st term $\mathcal{T}_c \sigma_3 = 1$ for all spin up.
1 spin flip: For 1 \rightarrow 2(d-1) plaquettes frustrated each neighboring plaquette can be thought of as a link on a dual lattice with 1 less dimension.

Each frustrated plaquette has relative energy 2, so 1 spin flip has relative energy 4(d-1). If N links on lattice and L links in contour,

\[ \langle \mathcal{T} \rangle = \langle \text{spin flip off lattice} \rangle \langle \text{spin flip on lattice} \rangle \]

\[ = \frac{1}{N^L} \exp \left[ -4\langle (d-1) \rangle \beta \right] \]

\[ = \frac{1}{N^L} \exp \left[ -4\langle L \rangle \beta \right] \]

\[ = \exp \left[ \frac{-4\langle L \rangle \beta}{N} \right] \]

For N spin flips, approximating them as completely independent should give:

\[ \frac{1}{N!} N^n \exp \left[ -4n(d-1) \beta \right] \]

This should be valid for \( N \gg n \).

Then:

\[ \langle \mathcal{T} \rangle = \sum_{n=0}^{\infty} \frac{1}{n!} (N-2L)^n \exp \left[ -4\langle L \rangle \beta \right] \]

\[ = \exp \left[ \frac{(N-2L)}{N} \exp \left[ -4\langle L \rangle \beta \right] \right] \]

\[ = \exp \left[ \frac{2L}{N} \exp \left[ -4\langle L \rangle \beta \right] \right] \]

If we don't assume completely independent spin flips,

\[ \langle \mathcal{T} \rangle \approx \exp \left[ -h(L) \right] \]

Penometer law.

So, high temp \( \rightarrow \) Area law

low temp \( \rightarrow \) Penometer law.

This low temp argument does not apply to 2D classical Ising gauge theory. Consider a line of spins flipped.

Only two frustrated plaquettes whatever is the length of the string, similar to how a domain in 1D classical Ising only pays energy penalty at its end, independent of domain length.

This tends to disorder the system at any nonzero T, leading to area law even in low T.
To see this, calculate the gauge invariant correlation functions, only taking into account spin configurations in which one end of the line extends to infinity, leaving a "tree" frustration.

Suppose one frustration near con-law C.

If outside C, string of flip spins affect even spins on C.

\[ + + \rightarrow + + + + \]

If inside C, string of flipped spins affect odd spins on C.

\[ \text{giving} \ -1 \]

If \( N \) is number of lines enclosed in C, then

\[ \langle T \sigma_3 \rangle = \frac{1 - (N - N_0) e^{-\beta} - N_0 e^{-\beta}}{1 - N_0 e^{-\beta}} \]

If we treat multiple frustrations as being independent, summing over their number gives

\[ \langle T \sigma_3 \rangle = \exp(-2 e^{-\beta} N_0) = \exp(-e^{-\beta} A) \]

Area law, even for low temp.

\( \langle T \sigma_3 \rangle \) does not detect any phase transition because there isn't 1.

Follows from the duality

\[ \begin{pmatrix} 2 \text{-dim} \\ (2 \text{-long gauge theory}) \end{pmatrix} = \begin{pmatrix} \text{One-d classical} \\ L \text{-long} \end{pmatrix} \]
Two-dim Ising gauge theory - 1d classical Ising

Fix temporal gauge, i.e. $G_{3}(n, rac{1}{2}) = 1$

temporal links.

Given a flipped temporal link, apply successive $G(n)$ to move it to infinity.

$$H = -J \sum_{x} \sigma_{x}^{3}(n, x) \sigma_{x}^{3}(n+1, x)$$

No coupling in spatial direction, only temporal direction.

Becomes a classical Ising 1D model, which is disordered for all $T > 0$.

Calculate $\langle T \bar{G}_{3} \rangle$.

$$T \bar{G}_{3} = \sigma_{3}(0, x) \sigma_{3}(0, x) \ldots \sigma_{3}(0, x)$$

$$\times \sigma_{3}(1, x) \sigma_{3}(1, x) \ldots \sigma_{3}(1, x)$$

These correlation functions are short ranged

$$\langle \sigma_{3}(n, 0; x) \sigma_{3}(n, 0; x) \rangle \sim \exp(-x)$$

So $\langle T \bar{G}_{3} \rangle \sim \exp(-\frac{R}{\xi}) = \exp(-\frac{R}{\xi}) = \exp(-\frac{R}{\xi})$

Get Area law again.
Recall the expansion:

\[ Z = \sum \prod (1 + 5 \cdot 5 \cdot 5 \cdot \tanh B) \]

\[ = 1 + \text{small} + \text{closed surfaces} \]

Low temp:

\[ 1 + \text{assign normal vector for frustrated plane} \]

Low classical Ising 3D

High Temp 3D

References

1) An introduction to lattice gauge theory and spin systems, John Q. Kogut.
2) Notes from Caltech Phys 127c Spring 2014, Olevei Motrunich
3) Statistical Physics of Fields, Mehran Kardar