Spinor Helicity Formalism / Franklin

There have been great developments of the on-shell scattering amplitudes in recent years. A lot of progress has been made on improving calculational techniques and gaining insights into the underlying mathematical structure.

Today, we provide an introduction to the spinor helicity formalism, as well as some elementary examples, which could serve as preparation for future speakers.

Consider a free Dirac field ψ $\quad \mathcal{L} = i\bar{\psi}\gamma_\mu \partial_\mu \psi - m\bar{\psi}\psi$

EOM $\Rightarrow$ $(i\not{\partial} + m)\psi = 0$. By plane wave expansion

$$\psi(x) \sim u(p)\, e^{ip\cdot x} + v(p)\, e^{-ip\cdot x} \qquad -p^2 = m^2$$

In momentum space, the EOM becomes

$$\begin{aligned} (\not{p} + m)\, u(p) &= 0 \\ (-\not{p} + m)\, v(p) &= 0 \end{aligned} \qquad (1)$$

Recall that $\not{p} = p_\mu \gamma^\mu = \begin{pmatrix} 0 & p_\mu \sigma^\mu \\ p_\mu \bar{\sigma}^\mu & 0 \end{pmatrix} = \begin{pmatrix} 0 & p_{a\dot{b}} \\ p^{\dot{a}b} & 0 \end{pmatrix}$

$\sigma^\mu = (I, \vec{\sigma}) \qquad \bar{\sigma}^\mu = (I, -\vec{\sigma})$

There is a distinction between the undotted and dotted spinor indices.

$SO(3,1) \approx SU(2) \times SU(2) \qquad \vec{A} = \frac{1}{2}(\vec{J} + i\vec{K}) \qquad \vec{B} = \frac{1}{2}(\vec{J} - i\vec{K})$

Dirac fermion is in $\underset{\alpha}{\overset{L\ \ R}{\left(\tfrac{1}{2}, 0\right)}} \oplus \underset{\dot{\beta}}{\overset{L\ \ R}{\left(0, \tfrac{1}{2}\right)}}$ $\qquad \begin{pmatrix} a \\ b \\ \dot{a} \\ \dot{b} \end{pmatrix}$

When $m = 0$, dotted and undotted components decouple.

(1) becomes $\not{p}\, v_\pm(p) = 0$, $\bar{u}_\pm(p)\, \not{p} = 0$ (2), with solution

$$v_+ = \begin{pmatrix} |p]_a \\ 0 \end{pmatrix}, \quad v_- = \begin{pmatrix} 0 \\ |p\rangle^{\dot{a}} \end{pmatrix}, \quad \bar{u}_-(p) = (0, \langle p|_{\dot{a}}), \quad \bar{u}_+(p) = ([p|^a, 0)$$

Raise and lower spinor indices $\quad [p|^a = \epsilon^{ab}|p]_b, \quad |p\rangle^{\dot{a}} = \epsilon^{\dot{a}\dot{b}}\langle p|_{\dot{b}}$

square : + $\qquad$ angle : −

It is one of the advantages of spinor helicity formalism that we do not need to find explicit reps for angle and square spinors. We can work abstractly with $|p\rangle$ and $|p]$, and later relate the results to the momentum vectors.

Two Eq.s in ② are related by Dirac conjugation if p is real.

Thus $[p|^a = \left(|p\rangle^{\dot a}\right)^*$ $\quad \langle p|_{\dot a} = \left(|p]_a\right)^*$, which allow us to know anti-chiral from chiral.

However, we would like to mention a completeness relation in spinor helicity notation $-\not p = |p\rangle[p| + |p]\langle p|$, more explicitly

$$\begin{pmatrix} 0 & -p_{a\dot b} \\ -p^{\dot a b} & 0 \end{pmatrix} = \begin{pmatrix} |p]_a \langle p|_{\dot b} & 0 \\ 0 & |p\rangle^{\dot a}[p|^b \end{pmatrix}$$

$v_\pm$ are already complete set of solutions to $\not p v = 0$.

$$\mathcal{O} = \sum_{\lambda_i} \lambda_i |i\rangle\langle i|$$

It's not a suprise that $p_{a\dot b}$ can be written as product of two vectors since for a 2×2 matrix, $\det p = 0 \iff -p_{a\dot b} = |p]_a \langle p|_{\dot b}$

$$\boxed{\begin{aligned} |p]_a &= \sqrt{2E}\begin{pmatrix} -\sin\frac{\theta}{2}\, e^{-i\phi} \\ \cos\frac{\theta}{2} \end{pmatrix} \\ |p\rangle^{\dot a} &= \sqrt{2E}\begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}\, e^{i\phi} \end{pmatrix} \end{aligned} \quad \text{properly normalized}}$$

It would be helpful to provide some properties of inner products of square and angle brakets. (strict math terminology?)

Define $\langle p q\rangle = \langle p|_{\dot a} |q\rangle^{\dot a}$ $\quad [pq] = [p|^a |q]_a$

$$\text{Metric} \sim \begin{pmatrix} \epsilon^{ab} & \\ & \epsilon_{\dot a \dot b} \end{pmatrix}$$

Due to the antisymmetry of Levi-Civitas,

$\langle pq\rangle = -\langle qp\rangle$, $[pq] = -[qp]$ Not exactly inner product!

All other brackets vanishes. $\langle p|q] = 0$

For real momentum $[pq]^* = \langle qp\rangle$

How are these brakets related to kinetics ?

$$\langle pq\rangle[pq] = -\langle pq\rangle[qp] = -\mathrm{Tr}\left(|p]\langle p|\;|q\rangle[q|\right)$$

$$= -\mathrm{Tr}\left(p_\mu \sigma^\mu q_\nu \bar\sigma^\nu\right) = 2\eta^{\mu\nu}p_\mu q_\nu = p\cdot q \underset{\text{massless + onshell}}{=} (p+q)^2$$

When defining angle-square braket $\langle p|\gamma^\mu|k]$, we have to be careful with the matching sigma matrix.

$\langle p|\gamma^\mu|k]$ is actually $\left(0, \langle p|\right)\begin{pmatrix} & \sigma^\mu \\ \bar\sigma^\mu & \end{pmatrix}\begin{pmatrix}|k] \\ 0\end{pmatrix}$

In this sense, the same helicity fermions lead to vanishing brackets

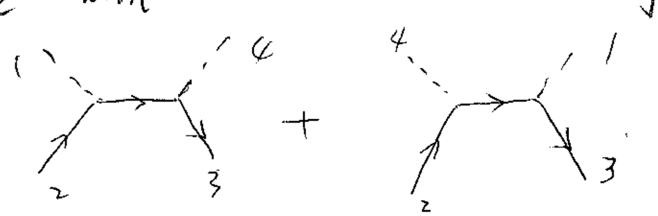
$$\langle p|\gamma^\mu|k\rangle = 0 = [p|\gamma^\mu|k]$$

Rmk: $\langle p| \sim \dot{2}$ $|k\rangle \sim \dot{2}$ γ^μ flip the helicity.

Fierz Identity $\langle 1|\gamma^\mu|2]\langle 3|\gamma_\mu|4] = 2\langle 13\rangle[24]$

Yukawa $\mathcal{L} = i\bar\psi\not\partial\psi - \frac{1}{2}(\partial\phi)^2 + g\phi\bar\psi\psi$

Ex: 4-point tree amplitude with 2 scalars and two fermions

$iA_4(\phi\,\bar f^{h_2} f^{h_3}\,\phi) =$ 

$$= (ig)^2\, \bar u_3 \frac{-i(\not p_1 + \not p_2)}{(p_1+p_2)^2} v_2 + (1 \leftrightarrow 4)$$

As mentioned above, we are interested in h_2, h_3 opposite.

$$A_4(\phi \bar{f}^+ f^- \phi) = g^2 \frac{\langle 3|\cancel{p_1}+\cancel{p_2}|2]}{(p_1+p_2)^2} + (1 \leftrightarrow 4)$$

$$= + \frac{\langle 3|\cancel{p_1}|2]}{(p_1+p_2)^2} + (1 \leftrightarrow 4)$$

$$= - g^2 \frac{\langle 31\rangle[12]}{\langle 12\rangle[12]} + (1 \leftrightarrow 4)$$

$$= g^2 \left(\frac{\langle 13\rangle}{\langle 12\rangle} + \frac{\langle 43\rangle}{\langle 42\rangle} \right)$$

$$A_4(\phi \bar{f}^- f^+ \phi) = g^2 \frac{[3|\cancel{p_1}+\cancel{p_3}|2\rangle}{(p_1+p_3)^2} + (1 \leftrightarrow 4)$$

$$= g^2 \frac{[3|\cancel{p_1}|2\rangle}{(p_1+p_3)^2} + (1 \leftrightarrow 4)$$

$$= - g^2 \frac{[31]\langle 12\rangle}{[31]\langle 31\rangle} + (1 \leftrightarrow 4)$$

$$= g^2 \frac{\langle 12\rangle}{\langle 13\rangle} + (1 \leftrightarrow 4)$$

Angle-square brakets have some properties like

$$[k|\gamma^\mu|p\rangle = \langle p|\gamma^\mu|k]$$

$$[k|\gamma^\mu|p\rangle^* = [p|\gamma^\mu|k\rangle$$

For light-like q

$$\langle p|\cancel{q}|k] \equiv \langle p|_{\dot a}\, q^{\dot a b}\, |k]_b = - \langle p|q\rangle \cdot [q|k]$$

$$= - \langle pq\rangle [qk]$$

Ex: QED

The polarization vectors are written as

$$\epsilon_-^\mu(p;q) = -\frac{\langle p|\gamma^\mu|q]}{\sqrt{2}\,[qp]}, \qquad \epsilon_+^\mu(p;q) = -\frac{\langle q|\gamma^\mu|p]}{\sqrt{2}\langle qp\rangle} \quad (3)$$

$$\therefore \quad \epsilon_-^\mu(p;q) = \epsilon_+^\mu(p;q)^* \quad \text{by def}$$

where $q \neq p$ denotes an arbitary <u>reference spinor.</u>

We can check the requirements $p_\mu \epsilon_\pm^\mu = 0$

$$p_\mu \epsilon_-^\mu(p;q) = -\frac{\langle p|\not{p}|q]}{\sqrt{2}\,[qp]} = -\frac{\langle pp\rangle[pq]}{\sqrt{2}\,[qp]} = \frac{1}{\sqrt{2}}\langle pp\rangle = 0$$

$$\epsilon_-^\mu \epsilon_{-\mu} = \frac{\langle p|\gamma^\mu|q]\langle p|\gamma_\mu|q]}{2\,[qp][qp]} = \frac{2\langle pp\rangle[qq]}{2\,[qp][qp]} = 0$$

$$\epsilon_+^\mu \epsilon_{+\mu} = 0$$

$$\epsilon_-^\mu \epsilon_{+\mu} = \frac{\langle p|\gamma^\mu|q]\langle q|\gamma^\mu|p]}{2\,[qp]\langle qp\rangle} = \frac{2\langle pq\rangle[qp]}{2[qp]\langle qp\rangle} = -1$$

It is useful to rewrite (3) as

$$\not{\epsilon}_-(p;q) = \frac{\sqrt{2}}{[qp]}\Big(|p\rangle[q| + |q]\langle p|\Big), \qquad \not{\epsilon}_+(p;q) = \frac{\sqrt{2}}{\langle qp\rangle}\Big(|p]\langle q| + |q\rangle[p|\Big)$$

One is free to shift $\epsilon_\pm^\mu(p) \to \epsilon_\pm^\mu(p) + C p^\mu$
because of gauge invariance. This leads to the arbitrariness of the choice of reference spinor.

To see this, let's look at $\not{\epsilon}_-(p,q) + \frac{\sqrt{2}}{[qp]} C\not{p}$

$$= \frac{\sqrt{2}}{[qp]}\Big(|p\rangle[q| + |q]\langle p| - C\,|p\rangle[p| - C\,|p]\langle p|\Big) = \frac{\sqrt{2}}{[\tilde{q}p]}\Big(|p\rangle[\tilde{q}| + |\tilde{q}]\langle p|\Big)$$

where $|\tilde{q}] = |q] - c|p]$ is a new ref. spinor

The simplest 3-particle QED process

$A_3(f^{h_1}\bar{f}^{h_2}\gamma^{h_3})$ (Though we know this process can never happen)

$$i A_3(f^-\bar{f}^+\gamma^-) = \bar{u}_-(p_1)\, ie\gamma_\mu\, v_+(p_2)\, \epsilon_-^\mu(p_3;q)$$
$$= -ie\langle 1|\gamma_\mu|2] \frac{\langle 3|\gamma^\mu|q]}{\sqrt{2}[3q]} = \sqrt{2}ie\frac{\langle 13\rangle[2q]}{[3q]}$$

The onshell amplitude should be independent of ref spinor q, which is not obvious.

$$\frac{\langle 13\rangle[2q]}{[3q]}\frac{\langle 12\rangle}{\langle 12\rangle}$$

$$\langle 12\rangle[2q] = -\langle 1|2|q]$$
$$= \langle 1|p_1+p_3|q] = \langle 1|3|q]$$
$$= -\langle 13\rangle[3q]$$

$$\therefore \quad A_3(f^-\bar{f}^+\gamma^-) = -\sqrt{2}e\,\frac{\langle 13\rangle^2}{\langle 12\rangle}$$

Or consider $A_4(\bar{f}^+ f^- \gamma^+\gamma^-)$ (2.58)