$\boxed{\text{Intro to CFT for EE}}$

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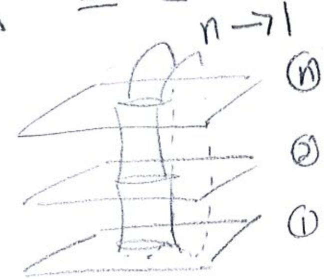
Outline

- Review + Motivation
- Conformal Group
- Operator Product Expansion
- Stress Tensor
- Primary Fields
- Conformal Ward Identity

Review + Motivation

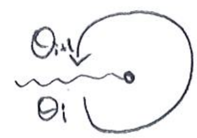
$$\rho_A = \mathrm{Tr}_B\, \rho \quad , \quad S_A = -\mathrm{Tr}\, \rho_A \ln \rho_A = -\lim_{n \to 1} \frac{\partial}{\partial n} \mathrm{Tr}\, \rho_A^n$$

Schematically $\mathrm{Tr}\, \rho_A^n = \int D\phi\, e^{-S_E}$ on



$\Longleftrightarrow \quad \mathbb{C}^{\otimes n} / \mathbb{Z}_n$

w/ $\sigma \theta_i = \theta_{i+1}$



CFT allows you to calculate and obtain $S_A = \frac{c}{3} \ln\left(\frac{l}{a}\right)$

This is done by finding $\mathrm{Tr}\, \rho_A^n \propto \langle \sigma_n(U), \bar{\sigma}_n(V) \rangle \propto \frac{1}{|U-V|^{2\Delta}}$

Using the transformation properties of the stress tensor and a conformal ward identity you can identify the scaling dimension.

Conformal Group

- Definition: A coordinate transformation $\xi \mapsto f(\xi)$ is conformal if $g_{\mu\nu} \longmapsto \rho(\xi)\, g_{\mu\nu}$, the metric changes by a factor
- In 2D this condition $\Rightarrow$ CRE $\Rightarrow$ $f(\xi)$ is analytic
- The power of complex analysis gives 2D CFT its power
- $Z \mapsto Z + \mathcal{E}(Z)$, $\mathcal{E}(z) = \sum_{n=-\infty}^{\infty} \epsilon_n z^{n+1}$
- infinitesimal generators $l_n = Z^{n+1}\frac{d}{dz}$, $e^{\epsilon_n l_n} Z = Z + \epsilon_n Z^{n+1}$
- satisfy $[l_n, l_m] = (n-m)\, l_{n+m}$
- translations, rotations, möbius transformations, scale transformations are subgroups

O.P.E.

- A useful concept in QFT, especially CFT is the Operator Product Expansion: $A_i(Z)A_j(0) = \sum C_{ij}^{k}(Z)A_k(0)$

- $C_{ij}^{k}(Z)$ are complex valued functions
- $\{A_k\}$ is some complete set of functions
- True inside correlation functions, $\langle A_i A_j X_1 \cdots X_N \rangle = \sum_k C_{ij}^{k} \langle A_k X_1 \cdots X_N \rangle$
- Why is it so useful in CFT? - Audience answers

Stress Tensor in CFT

- If $\xi^\mu \to \xi^\mu + \epsilon^\mu(\xi)$, $\sum_{k=1}^{N} \langle A_1(\xi_1) \cdots \delta_\epsilon A_k(\xi_k) \cdots A_N(\xi_N) \rangle$
$= -\int d^2\xi\, \partial^\mu \epsilon^\nu(\xi) \langle T_{\mu\nu}(\xi) A_1 \cdots A_N \rangle$, $T_{\mu\nu} = \frac{1}{\sqrt{g}} \frac{\delta S}{\delta g^{\mu\nu}}$

- For a theory to be scale invariant we must have $T_\mu{}^\mu = 0$ in flat space.

- Generally consider holomorphic (actually meromorphic) $T(z)$ and anti-holomorphic $\bar{T}(\bar{z})$ parts of stress tensor.

- Expand $T(z) = \sum \frac{L_n}{z^{n+2}}$, $L_n = \oint \frac{dz}{2\pi i} z^{n+1} T(z)$

- coefficients $\{L_n\}$ form the Virasoro algebra

- OPE of stress tensor with itself:

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2}{(z-2)^2}T(w) + \frac{1}{z-w}\partial_w T(w)$$

$$+ L_{-2}T(w) + (z-w) + \cdots$$

- Infinitesimal form $\delta_\epsilon T(w) = \oint_w dz\, \varepsilon(z) T(z) T(w)$

$$= [\varepsilon(w)\partial_w + 2\partial_w]T(w) + \frac{c}{12}\partial_w^3 \varepsilon(w)$$

$$\Rightarrow T(z) \to T(w) - \left(\frac{dz}{dw}\right)^2 T(z) + \frac{c}{12}\{z, w\} \leftarrow \text{Important}$$

where $\{z, w\} = (z'z''' - \frac{3}{2}z''^2)/z'^2$

Primary Fields

- $\Phi(z,\bar{z})$ is called primary if $\Phi(w,\bar{w}) = \Phi(z,\bar{z})\left(\frac{dz}{dw}\right)^h \left(\frac{d\bar{z}}{d\bar{w}}\right)^{\bar{h}}$
- where $h, \bar{h}$ are the eigenvalues of $L_0, \bar{L}_0$
- $\Delta = h + \bar{h}$ is called "scaling dimension", $s = h - \bar{h}$ is called spin
- infinitesimal form is $\delta_\epsilon \Phi(z) = [h\partial\epsilon + \epsilon\partial]\Phi(z)$

- Conformal invariance fixes: $\langle \Phi_1(z_1)\Phi_2(z_2)\rangle = \dfrac{C_{12}}{|z_{12}|^{2\Delta}}\delta_{h_1,h_2}$

- Can argue on general grounds or by using $\delta_\epsilon\langle \Phi_1(z_1)\,\Phi_2(z_2)\rangle = 0$ and solving the differential eq. setting $\epsilon = 1, z, z^2$

$\boxed{\text{Conformal Ward Identity}}$

- $T(z)\Phi(w) = \dfrac{h}{(z-w)^2}\Phi(w) + \dfrac{1}{z-w}\partial_w\Phi(w) + \Phi^{-2}(w)$

$+ (z-w)\Phi^{-3}(w) + \cdots$

- Fields Φ^{-n} are called secondaries or descendants of $\Phi(w)$

- $\Phi^{-n}(w) = L_{-n}\Phi(w) = \oint \dfrac{dz}{2\pi i}(z-w)^{-n+1}\,T(z)\Phi(w)$

- We see that $L_0\Phi(z) = h\Phi(z)$, $L_{-1}\Phi(z) = \partial_z\Phi(z)$

  $L_n\Phi(z) = 0$, $n > 0$.

- Primary fields are lowest weight states

- OPE can be used to evaluate $\langle T(z)\Phi_1(w_1)\cdots\Phi_n(w_n)\rangle$

$= \displaystyle\sum_{j=1}^{N}\left[\frac{h_j}{(z-w_j)^2} + \frac{1}{z-w_j}\frac{\partial}{\partial w_j}\right]\langle\Phi_1(w_1)\cdots\Phi_N(w_N)\rangle$

- called Conformal Ward Identity

– The End –