Holographic Quantum Error Correction

Overview

Almheiri, Dong, Harlow 1411.7041

- Operator Algebra Quantum Error Correction.
  - a simple example of "
  - precise statement of "
- Operator reconstruction in AdS/CFT
  - as a realization of quantum error correction

Pastawski, Yoshida, Harlow, Preskill 1503.06237

- Exactly solveable lattice realization

Operator Algebra Quantum Error Correction

Jingyuan + I will be solving diff. problems

- JY: how do I ensure I can retrieve a desired state after a given no. of errors
- Me: how do I ensure I can perform a given logical operation after errors?
  - only concerned w/ one type of error: erasure.
- Basic idea:
  - Alice has some k-qubit quantum state $|\psi\rangle \in \mathcal{H}^k$
  - Bob has a quantum operation O he wants to perform on Alice's information.
  - Worried that while in transit, l-qubits could be lost.
    - can no longer perform his computation O w/ only $k-l$ qubits since O acts on full $\mathcal{H}^k$.
- Plan: Alice encodes her data in a redundant way using $n>k$ qubits.

$$U_{code}: \mathcal{H}^k \to \mathcal{H}^n$$

where U_{code} is an isometry: $\langle \psi, \phi\rangle = \langle U_{code}\psi, U_{code}\phi\rangle$

- U_{code} is called an error correcting code

$\mathcal{H}^k$ logical hilbert space

$\mathcal{H}^n$ physical hilbert space

$\mathcal{H}_c = \mathrm{Im}\, U_{code}$ code subspace. ($U_{code}: \mathcal{H}^k \to \mathcal{H}_c$ is unitary)

- Since encoding is redundant, can loose some qubits + still carry out the computation:

Suppose $O: \mathcal{H}^k \to \mathcal{H}^k$ is operation Bob wants to perform. (logical operator)

$\tilde{O} = U_{code} O U_{code}^\dagger : \mathcal{H}_c \to \mathcal{H}_c$ is the physical operator.

- the point is, there are many operators on $\mathcal{H}^n$ that reduce to $\tilde{O}$ on $\mathcal{H}_c$.
- in particular given certain assumptions that we'll discuss later, if loose a set E of l spins, $\exists$ operator $\tilde{O}_E$ on the remaining $n-l$ st.

$$\tilde{O}_E |\tilde{\psi}\rangle = \tilde{O}|\tilde{\psi}\rangle \quad \text{for all } |\tilde{\psi}\rangle \in \mathcal{H}_c$$

and Bob can compute away.

- Bob then has many different realizations of the operator O that he can use depending on which set of spins that can be erased
  - called <u>operator algebra quantum error correction</u>
- this is the sense in which AdS/CFT realizes error correction.
  - many possible realizations of a bulk operator $\phi(x)$ when viewed in the boundary hilbert space
  - Under erasure of part of the boundary, $\exists$ a realization of $\phi(x)$ on its complement (how much can be erased depends on how far $\phi(x)$ is into the bulk).

3-qubit example

- Alice wants to send state
$$|\psi\rangle = \sum_{i=0}^{2} a_i |i\rangle$$
(not $\mathcal{H}$ is 3-dimensional, but we're still going to call it a qubit)
- To protect against erasure she encodes it in a 3-qubit state
$$|\tilde{\psi}\rangle = \sum_{i=0}^{2} a_i |\tilde{i}\rangle \quad \text{where}$$
$$\left.\begin{aligned} |\tilde{0}\rangle &= \frac{1}{\sqrt{3}}\left(|000\rangle + |111\rangle + |222\rangle\right) \\ |\tilde{1}\rangle &= \frac{1}{\sqrt{3}}\left(|012\rangle + |120\rangle + |201\rangle\right) \\ |\tilde{2}\rangle &= \frac{1}{\sqrt{3}}\left(|021\rangle + |102\rangle + |210\rangle\right) \end{aligned}\right\} \text{ span code subspace of } \mathcal{H}^3$$
- for any $|\tilde{\psi}\rangle \in \mathcal{H}_c$ any one of the qubits is in a maximally entangled state
$$\frac{1}{3}\left(|0\rangle\langle 0| + |1\rangle\langle 1| + |2\rangle\langle 2|\right)$$
so can't actually be losing information if one qubit is lost.
- indeed if Bob gets two qubits he actually has access to all the information
- he can apply a unitary transformation to the two qubits he receives.

$$U_{12}: \quad \begin{array}{lll} |00\rangle \to |00\rangle & |11\rangle \to |01\rangle & |22\rangle \to |02\rangle \\ |01\rangle \to |12\rangle & |12\rangle \to |10\rangle & |20\rangle \to |11\rangle \\ |02\rangle \to |21\rangle & |10\rangle \to |22\rangle & |21\rangle \to |20\rangle \end{array}$$

$$(U_{12} \otimes I_3)|\tilde{\psi}\rangle = |\psi\rangle \otimes \frac{1}{\sqrt{3}}\left(|00\rangle + |11\rangle + |22\rangle\right)$$

- can now perform his logical operation O and transform back

$$\tilde{O}_{12} = U_{12}^{\dagger} O U_{12}$$

this satisfies $\tilde{O}_{12}|\tilde{\psi}\rangle = \tilde{O}|\tilde{\psi}\rangle$

and only acts on the two qubits he has in his possession!

can similarly define $\tilde{O}_{23}$ & $\tilde{O}_{13}$ s.t.

$$\tilde{O}_{12}|\tilde{\psi}\rangle = \tilde{O}_{23}|\tilde{\psi}\rangle = \tilde{O}_{13}|\tilde{\psi}\rangle = \tilde{O}|\tilde{\psi}\rangle$$

- Bob can always perform the computation if any 1 qubit is lost
- We say the protocol corrects against erasure of a single qubit.

When is this possible in general?

<u>Theorem</u> Consider an operator $O: \mathcal{H}_c \to \mathcal{H}_c$ on a code subspace $\mathcal{H}_c \subseteq \mathcal{H}_{\bar{E}} \otimes \mathcal{H}_E$. Then there exists $O_{\bar{E}}: \mathcal{H}_{\bar{E}} \to \mathcal{H}_{\bar{E}}$ such that

$$O_{\bar{E}}|\psi\rangle = O|\psi\rangle \qquad O_{\bar{E}}^{\dagger}|\psi\rangle = O|\psi\rangle \qquad \text{for all } |\psi\rangle \in \mathcal{H}_c$$

iff O commutes w/ all operators $X_E: \mathcal{H}_E \to \mathcal{H}_E$ on the code subspace.

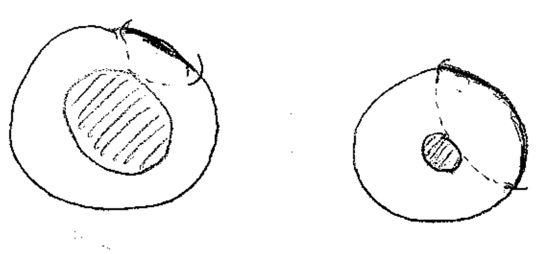
$$\langle\phi|[O, X_E]|\psi\rangle = 0 \qquad \forall |\psi\rangle, |\phi\rangle \in \mathcal{H}_c.$$

<u>proof:</u> See appendix B of 1411.7041
or the original papers

Bény, Kempf, Kribs "Generalization of quantum error correction via the heisenberg picture" PRL 98 (2007)

& "Quantum error correction of observables" PRA 76 (2007)

Note: 1) Set of reconstructable O forms an algebra
2) For every possible set of erasures E, there is a different algebra I can reconstruct
3) If increase size of erasure E, reconstructable algebra reduces to a subalgebra.
   - this is something we'll see in AdS/CFT



if allow erasure of larger & larger boundary regions, reconstructable algebra of local bulk operators lies deeper & deeper in the bulk.

Operator Reconstruction in AdS/CFT

original refs: Hamilton, Kabat, Lifschytz, Lowe hep-th/0606141

caveats: · low dim. so doesn't lead to significant backreaction
· I'm imagining a supergrav. operator within the framework of QFT on a curved background space time.

SAVE

Claim: If $\phi(x)$ is a bulk local operator within the "causal wedge" of some boundary spatial region A, there is a boundary expression for $\phi(x)$ that involves only operators on A:

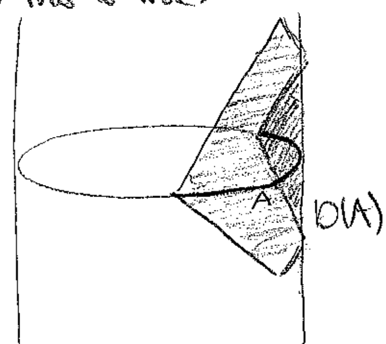
$$\phi(x) = \int_A dy\, K(x;y)\, O(y).$$

$K(x,y)$ is called a "smearing function".

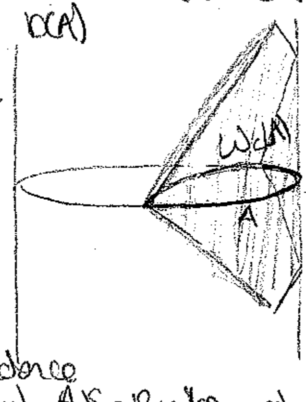
more caveats: · proof is for free fields on AdS space.
· apparently there is a way to make a systematic expansion in $\frac{1}{N}$ (interactions?)
· status is unknown on spacetimes that are merely asymptotically AdS.

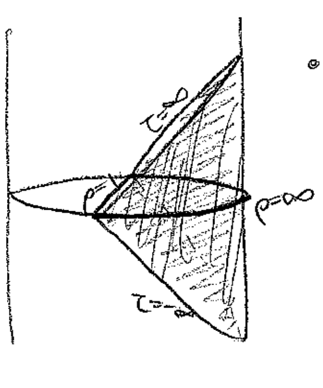
· first I'll define the causal wedge, then I'll show this is true.

· For a spatial region A let $D(A)$ be it's domain of dependence
(set of all points so that either all past or all future extendible causal curve passes through A)



· The causal wedge $W_c(A)$ is the set of all bulk points that can be reached by both a future directed causal curve from $D(A)$ and a past directed causal curve from $D(A)$
· Not really important how this comes about but it's sufficiently simple, let's do it anyways.
· To show the claim we will consider the case where A is a hemisphere of S^{d-1} (the boundary spatial slice.) (all other discs conformally equivalent.





· same as domain of dependence of half-AdS space, called AdS-Rindler wedge

coordinates

$$ds^2 = -(\rho^2-1)d\tau^2 + \frac{d\rho^2}{\rho^2-1} + \rho^2 d\Omega^2_{d-1}$$

metric on hyperbolic disc A.

☆ why is it hyperbolic? isn't it just flat hemisphere?

- $W_C[A]$ is globally hyperbolic so can do QFT on curved spacetime!

  decompose $\phi(x)$ into raising + lowering ops.
  $$\phi(\rho,\tau,\Omega) = \int_0^\infty \frac{d\omega}{2\pi} \sum_\lambda \left(f_{\omega\lambda}(\rho,\tau,\Omega) a_{\omega\lambda} + f^*_{\omega\lambda}(\rho,\tau,\Omega) a^\dagger_{\omega\lambda} \right) \quad (*)$$

  - $f_{\omega\lambda}$ solutions to EOM w/ BC $\quad f_{\omega\lambda} \sim \rho^{-\Delta} e^{-i\omega\tau} Y_\lambda(\Omega)$ as $\rho \to \infty$.
    λ eigenvalue of S^{d-1} laplacian. $\quad \Delta = \frac{d}{2} + \frac{1}{2}\sqrt{d^2+4m^2}$

  - AdS/CFT dictionary:
    $$\lim_{\rho\to\infty} \rho^\Delta \phi(\rho,\tau,\Omega) = \underbrace{O(\tau,\Omega)}_{\text{op of dim } \Delta}$$
    $$\text{so} \quad O(\tau,\Omega) = \int_0^\infty \frac{d\omega}{2\pi} \sum_\lambda \left(N_{\omega\lambda} e^{-i\omega\tau} Y_\lambda(\Omega) a_{\omega\lambda} + \text{c.c.} \right).$$

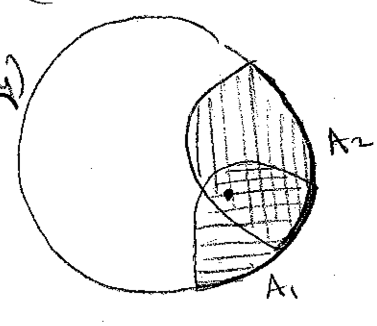
3.14 3.19 3.21. approximate: 3.23

$$\text{invert:} \quad a_{\omega\lambda} = \frac{1}{N_{\omega\lambda}} \int d\tau\, d\Omega\, e^{i\omega\tau} Y^*_\lambda(\Omega) O(\tau,\Omega)$$

finally plug back into $(*)$:
$$\phi(\rho,\tau,\Omega) = \int_{D(A)} d\tau' d\Omega'\, K(\rho,\tau,\Omega;\tau',\Omega')\, O(\tau',\Omega')$$
$\left(e^{iH\tau'} O_s(\Omega') e^{-iH\tau'}\right) = O(\tau',\Omega')$ operators on A.

$$\text{where} \quad K(\rho,\tau,\Omega;\tau',\Omega') = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \sum_\lambda \frac{1}{N_{\omega\lambda}} f_{\omega\lambda}(\rho,\tau,\Omega) e^{-i\omega\tau'} Y^*_\lambda(\Omega').$$

<u>Key Feature</u>: • $\phi(x)$ lies in many different causal wedges.
- can represent $\phi(x)$ as a boundary operator w/ support on different regions
$$\phi(x) = \int_{A_1} dy\, K_{A_1}(x,y) O(y) = \int_{A_2} dy\, K_{A_2}(x,y) O(y)$$



- looks a lot like OAQEC:
  $\phi(x) \in W_C(A)$ as a CFT operator is protected against erasure of $\bar{A}$.

Let's try to make this analogy as precise as possible by phrasing it in the same language of the earlier theorem.

- Define code subspace $\mathcal{H}_c$ of the CFT hilbert space.

$$\mathcal{H}_c = \text{Span}\{|\Omega\rangle, \phi_i(x)|\Omega\rangle, \phi_i(x_1)\phi_j(x_2)|\Omega\rangle, \dots\} \subseteq \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$$

for a finite selection of bulk operators $\phi_i(x)$

For definiteness we are realizing $\phi_i(x)$ in the CFT via the global reconstruction

$$\phi_i(x) = \int_{S^{d-1}} K(x;y) O_i(y)$$

↑ take A to be the entire spatial slice in the above construction.

- the $\phi_i(x)$ are the logical operators on $\mathcal{H}_c$ we will try to realize (really we mean their projection onto $\mathcal{H}_c$, but apparently these do not affect low-point correlators (low being $\ll N$) so we can forget them - see footnote 13 - I don't understand this)

- Now by bulk causality operators on $W_c(A)$ commute w/ those on $W_c(\bar{A})$

↑ operators $X_{\bar{A}}$ that can be realized purely on $\bar{A} = E$

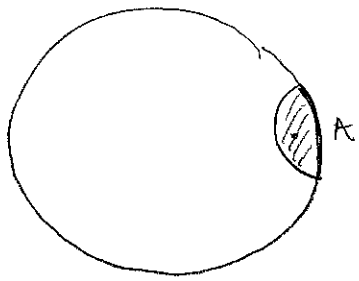
- Thus we are in the situation of the theorem

$$[O, X_{\bar{A}}] = 0 \qquad \text{where } O \in \{\phi_i(x)\} \text{ \& } X_{\bar{A}} \text{ have support on } \bar{A}$$

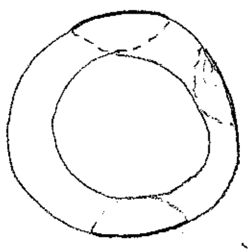
& the O's have a realization w/ support on A.

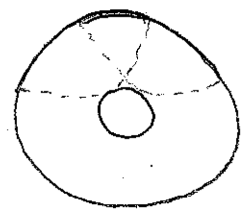
caveat: have not proven directly for all operators $X_{\bar{A}}$ on $\bar{A}$, only the subset that are realizations of local operators on $W_c(\bar{A})$.

- Note: can never realize the full bulk operator algebra.
  If allow erasure of A, cannot realize bulk operators that live in $W_C(A)$



- So AdS-Rindler reconstruction is many OA error correcting codes at once.
  If allow erasure of certain percentage of boundary, can only realize algebra sufficiently far from boundary





If allow larger regions of erasure
the reconstructable algebra lies deeper in bulk.

- So in this picture, radial position encodes how well protected an operator is against erasure

- most protected operators are at the center, which are protected against erasure of half the boundary

  (never mind that there is no special "center" point in AdS -
  I don't understand this)