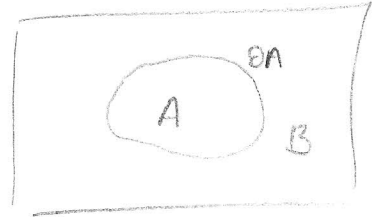


Area Laws for Entanglement Entropy

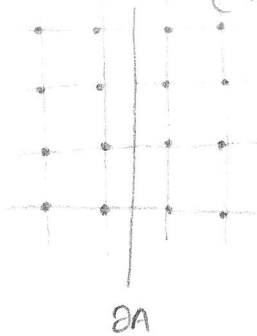
- Main point: for large class of systems
- ① EE between regions A+B in ground state
 $\sim \frac{\text{area of boundary}}{\text{cutoff}^{d-1}}$ d spatial dimensions.



$$S_A = -\text{Tr} \rho_A \ln \rho_A$$

- Can be understood heuristically in a few ways

- since $S_A = S_B$ at least cannot be extensive.
 - must depend only on common feature, such as ∂A
- if correlations are short length



- little entanglement between far separated spins
- most entanglement concentrated on boundary where have closest spins.

- Mirrors black hole entropy $S_{\text{BH}} = \frac{1}{4} M_{\text{pl}}^2 A$ A area of horizon.

- ② -idea: no access to within horizon, so external observer views mixed state that is partial trace \Rightarrow entanglement entropy

⊛ How seriously should I be taking this?

◦ Lets make these ideas more precise

Outline • $\Sigma\Sigma$ for 2 coupled oscillators.

- Outline of Srednicki's calculation
 - free massless KG field
 - discretize, find $\underbrace{\text{finite}}$ collection of coupled oscillators
- Path integral formulation
 - n -sheeted riemann surfaces
- Area laws via path integrals
- Exceptions.

2 coupled oscillators

• a la Srednicki, start w/ easy 2-oscillator problem & generalize

$$H = \frac{1}{2} (p_1^2 + p_2^2 + k_0(x_1^2 + x_2^2) + k_1(x_1 - x_2)^2)$$

• seek entanglement between spins 1 & 2 in ground state.

• so let's get the ground state!

- diagonalize coupling matrix

$$H = \frac{1}{2} (p_+^2 + p_-^2 + \omega_+^2 x_+^2 + \omega_-^2 x_-^2)$$

where $x_{\pm} = (x_1 \pm x_2) \frac{1}{\sqrt{2}}$
 $\omega_+ = k_0^{\frac{1}{2}}$ $\omega_- = (k_0 + 2k_1)^{\frac{1}{2}}$

- ground state product of single particle ground states

$$\begin{aligned} \psi_0(x_1, x_2) &\propto \exp\left(-\frac{1}{2}(\omega_+ x_+^2 + \omega_- x_-^2)\right) \\ &= \exp\left(-\frac{1}{4}(\omega_+ + \omega_-)(x_1^2 + x_2^2) - \frac{1}{2}(\omega_+ - \omega_-)x_1 x_2\right) \end{aligned}$$

• ground state density matrix: $\langle \vec{x}' | \rho | \vec{x} \rangle = \psi_0(x_1, x_2) \psi_0^*(x'_1, x'_2)$

- take $A = \text{spin } 1$
 $B = \text{spin } 2$

$$\langle x_1 | p_A | x_1' \rangle = \int dx_2 \psi_0(x_1, x_2) \psi_0^*(x_1', x_2) \propto \exp \left\{ -\gamma(x_1^2 + x_1'^2)/2 + \beta x_1 x_1' \right\}$$

↑
gaussian integral

$$\text{where } \beta = \frac{1}{4} \frac{(w_+ - w_-)^2}{w_+ + w_-} \quad \gamma - \beta = \frac{2w_+ w_-}{w_+ + w_-}$$

- Now $S_A = -\sum_n p_n \ln p_n$ where p_n are eigenvalues of above operator.

$$\int dx' p_A(x, x') f_n(x') = p_n f(x)$$

- p_A looks like SHO propagator

$$\text{so guess } f_n = H_n(\alpha x) \exp(-\frac{1}{2}\alpha x^2) \quad \alpha = (\gamma^2 - \beta^2)^{\frac{1}{2}}$$

$$\Rightarrow p_n = (1 - \xi) \xi^n \quad \text{where } \xi = \frac{\beta'}{1 + (1 - \beta'^2)^{\frac{1}{2}}} \quad \beta' = \frac{\beta}{\gamma}$$

$$S_A = -\sum_n (1 - \xi) \xi^n \ln(1 - \xi) \xi^n = -\sum_n (1 - \xi) \ln(1 - \xi) \xi^n - \sum_n (1 - \xi) \ln \xi^n \xi^n$$

$$= -\ln(1 - \xi) - \frac{\xi}{1 - \xi} \ln \xi \quad \text{where we've used } \sum \xi^n = \frac{1}{1 - \xi} \quad \sum n \xi^n = \frac{\xi}{(1 - \xi)^2}$$

- Can generalize to N oscillators (just gaussian integrals!)

$$H = \frac{1}{2} \sum_{i=1}^N p_i^2 + \frac{1}{2} \sum_{i,j=1}^N x_i K_{ij} x_j \quad \text{let } K^{\frac{1}{2}} = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$$

if take $A = 1, \dots, n \quad B = n+1, \dots, N$

$$S_A = -\sum_{i=1}^n \left(\ln(1 - \xi_i) + \frac{\xi_i}{1 - \xi_i} \ln \xi_i \right) \quad \text{where } \xi_i = \frac{\beta_i'}{1 + (1 - \beta_i'^2)^{\frac{1}{2}}}$$

β_i' are eigenvalues of

$$\beta' = \frac{1}{2} \left(C - \frac{1}{2} B^T A^{-1} B \right)^{\frac{1}{2}} (B^T A^{-1} B) \left(C - \frac{1}{2} B^T A^{-1} B \right)^{\frac{1}{2}}$$

