A Very Introductory AdS/CFT

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The AdS/CFT correspondence, conjectured by Maldecena in late 1997, has been playing an essential role in modern string theory. The original version of this gravity/gauge AdS/CFT correspondence states an equivalence between Type 2B string theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ super Yang-Mills theory in 4D, under certain parameter region. Namely, for these two theories, the operator variables, physical states, correlation functions and dynamics, are equivalent to one another.

In this elementary-level note, we briefly discuss the geometry of AdS space, and show that the isometry group of AdS_{d+1} is isomorphic to the conformal group of the flat Minkowski space $\mathbb{R}^{1,d-1}$, which motivates the AdS/CFT. Then we will provide a precise statement of the AdS/CFT correspondence. We mainly follow the expositions of following references:

1. Aharony, O., Gubser, S.S., Maldacena, J., Ooguri, H. and Oz, Y., 2000. Large N field theories, string theory and gravity. Physics Reports, 323(3), pp.183-386. ArXiv: 9905111.

2. D'Hoker, Eric, and Daniel Z. Freedman. "Supersymmetric gauge theories and the AdS/CFT correspondence." ArXiv:0201253v2.

1 Anti-de Sitter space

The $(d+1)$-D AdS can be represented by hyperboloid

$$X_0^2 + X_{d+1}^2 - \sum_{i=1}^{d} X_i^2 = R^2 \tag{1}$$

embedded in the flat $(d+2)$-D space with the intrinsic metric

$$ds^2 = -dX_0^2 - dX_{d+1}^2 + \sum_{i=1}^{d} dX_i^2 \; . \tag{2}$$

We can think of the flat $(d+2)$-D space has two timelike directions, and Eq.(1) defines the Minkowskian AdS_{d+1}. The constructed AdS_{d+1} has isometry $SO(2,d)$, and is isotropic. We can solve Eq.(1) by setting

$$X_0 = R\cosh\rho\cos\tau, \tag{3a}$$

$$X_{d+1} = R\cosh\rho\sin\tau, \tag{3b}$$

$$X_i = R\sinh\rho\,\Omega_i, \quad \sum_{i=1}^{d}\Omega_i^2 = 1\ , \tag{3c}$$

and obtain the metric of AdS_{d+1}

$$ds^2 = R^2(-\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho\, d\Omega^2)\ . \tag{4}$$

The solution Eq.(3a) covers the entire hyperboloid once with $(\rho \geq 0, 0 \leq \tau \leq 2\pi)$. Therefore, the coordinates (τ, ρ, Ω_i) are called the global coordinates of AdS, and the hyperboloid has the topology of $S^1 \times \mathbb{R}^d$.(It can be shown by taking $\rho \to 0$, and keeping the leading order in each term of the metric.) To obtain a causal spacetime, we can wrap the closed τ-circle S^1 and take the universal covering of the hyperboloid. The isometry group $SO(2,d)$ of AdS_{d+1} has the maximal compact subgroup $SO(2) \times SO(d)$. From the new coordinates (τ, ρ, Ω_i), we find the $SO(2)$ part corresponds to the constant translation in the τ direction, and the $SO(d)$ gives the rotations of S^{d-1}, obtained by $\sum_{i=1}^{d}\Omega_i^2 = 1$.

To study the causal structure of AdS_{d+1}, we may transform the coordinate ρ into θ by

$$\sinh\rho = \tan\theta, \qquad 0 \leq \theta < \pi/2, \tag{5}$$

and now the metric reads

$$ds^2 = \frac{R^2}{\cos^2\theta}(-d\tau^2 + d\theta^2 + \sin^2\theta d\Omega^2). \tag{6}$$

After a conformal rescaling, we obtain the same metric as the Einstein static universe. However, it is only one-half of the Einstein static universe, because θ takes values in $[0, \pi/2)$ rather than $[0, \pi)$. In general, we call a spacetime **Asympototically** AdS, if it can be conformally compactified into a region which has the same boundary structure as one-half of the Einstein static universe. While $\theta = \pi/2$ is the boundary of the space with the topology of S^{d-1}.

Thus the boundary of the conformal compactified AdS_{d+1} is identical to the conformal compactification of the d-D Minkowski space, i.e., $\mathbb{R} \times S^{d-1}$. This provides another motivation for AdS_{d+1}/CFT_d correspondence.

Remark 1. *The d-D Minkowski space $\mathbb{R}^{1,d-1}$ is conformally compactified to $\mathbb{R} \times S^{d-1}$. That is, only spatial dimensions undergo the stereographic mapping.*

There is another convenient set of coordinates, the Poincare coordinates $(u, t, \boldsymbol{x})$ $(u > 0, \boldsymbol{x} \in \mathbb{R}^{d-1})$, which is parameterized by

$$X_0 = \frac{1}{2u}\left(1 + u^2(R^2 + \boldsymbol{x}^2 - t^2)\right), \tag{7a}$$

$$X_i = Rux_i, \tag{7b}$$

$$X_d = \frac{1}{2u}\left(1 - u^2(R^2 - \boldsymbol{x}^2 + t^2)\right), \tag{7c}$$

$$X_{d+1} = Rut\ . \tag{7d}$$

These coordinates cover one-half of the hyperboloid Eq.(1), and lead to another form of the AdS_{d+1} metric

$$ds^2 = R^2 \left(\frac{du^2}{u^2} + u^2(-dt^2 + d\boldsymbol{x}^2) \right) \ . \tag{8}$$

It's worth noting that Eq.(8) enjoys the manifest isometry subgroup $ISO(1, d-1)$ and $SO(1,1)$. And the $SO(1,1)$ acts as

$$(u, t, \boldsymbol{x}) \to (\lambda^{-1}u, \lambda t, \lambda \boldsymbol{x}), \quad \lambda > 0 \ , \tag{9}$$

and this identified with the dilatation D in the conformal group of $\mathbb{R}^{1,d-1}$.

Since the metric Eq.(4) is static w.r.t the global time coordinate τ, Wick rotation is allowed for the QFT on AdS_{d+1}. In the coordinates (τ_E, ρ, Ω_i) and $(u, t_E, \boldsymbol{x})$, the Euclidean metric is expressed as

$$\tau \to \tau_E = -i\tau, \quad ds_E^2 = R^2(\cosh^2 \rho d\tau_E^2 + d\rho^2 + \sinh^2 \rho \, d\Omega^2) \ ; \tag{10a}$$

$$t \to t_E = -it, \quad ds_E^2 = R^2 \left(\frac{du^2}{u^2} + u^2(dt_E^2 + d\boldsymbol{x}^2) \right) \ . \tag{10b}$$

Remark 2. *The Euclidean metric Eq.(10b) gives the hyperbolic geometry, if we write* $u = 1/y$

$$ds_E^2 = R^2 \frac{dy^2 + dt_E^2 + d\boldsymbol{x}^2}{y^2} \ . \tag{11}$$

As with the hyperbolic plane, AdS *is curved in such a way that any point in the interior is actually infinitely far from this boundary surface.*

The Euclidean AdS_{d+1} can be mapped into a $(d+1)$-D disk. In the coordinates $(u, t_E, \boldsymbol{x})$, $u \to \infty$ $(y \to 0)$ represents the sphere S^d at the boundary with one point removed, and the full boundary sphere is recovered by adding the point $u = 0$.

2 Conformal structure of flat space and conformal group

The conformal group is a group of transformations which preserve the form of the metric up to a scale factor,

$$g_{\mu\nu}(x) \to \Omega^2(x) g_{\mu\nu}(x) \ . \tag{12}$$

The direct result of a conformal map is that the angle between two vector fields are preserved pointwise. Apart from the generators in the Poincare group, the conformal group of Minkowski space also contains the dilatation

$$D : x^\mu \to \lambda x^\mu \ , \tag{13}$$

and the special conformal transformation

$$K^\mu : x^\mu / \boldsymbol{x}^2 \to x^\mu / \boldsymbol{x}^2 - b^\mu \ , \tag{14}$$

which can be thought of as an inversion combined with translation.

We begin with the Minkowski space $\mathbb{R}^{1,d-1}$:

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega_{d-2}^2 . \tag{15}$$

Applying a series of coordinate transformation,

$$u_\pm = t \pm r \tag{16a}$$
$$\tan \tilde{u}_\pm = u_\pm \tag{16b}$$
$$(\tau \pm \theta)/2 = \tilde{u}_\pm \,, \tag{16c}$$

we rewrite the metric as

$$ds^2 = \frac{1}{4\cos^2 \tilde{u}_+ \cos^2 \tilde{u}_-}(-d\tau^2 + d\theta^2 + \sin^2\theta \, d\Omega_{d-2}^2) \,. \tag{17}$$

The conformal rescaled metric can be analytically continued outside the trangular region of the (τ, θ) plane to

$$0 \le \theta \le \pi, \quad \tau \in \mathbb{R}. \tag{18}$$

The maximally extended space pertains the geometry of $\mathbb{R} \times S^{d-1}$ (Einstein static universe). The global conformal group on $\mathbb{R}^{1,d-1}$ is $SO(2,d)$, which is an enlargement of the Lorentz group $SO(1,d-1)$. The generators of $SO(2,d)$ are identified with the Poincare generators P_μ and $M_{\mu\nu}$, the dilatation D, and the special conformal generators K_μ by

$$J_{\mu\nu} = M_{\mu\nu}, \tag{19a}$$
$$J_{\mu,d} = \frac{1}{2}(K_\mu - P_\mu), \tag{19b}$$
$$J_{\mu,d+1} = \frac{1}{2}(K_\mu + P_\mu), \tag{19c}$$
$$J_{d+1,d} = D. \tag{19d}$$

These generators obey the commutation relations

$$[J_{ab}, J_{cd}] = -i(g_{ac}J_{bd} - g_{bc}J_{ad} + g_{bd}J_{ac} - g_{ad}J_{bc}) \tag{20}$$

with the flat metric

$$g_{ab} = \text{diag}(-1, 1, 1, \cdots, 1, -1).$$

After a Wick rotation,

$$\mathbb{R}^{1,d-1} \to \mathbb{R}^d. \tag{21}$$

Since $\mathbb{R}^d$ is conformally equivalent to S^d, the field theory on $\mathbb{R}^d$ is equivalent to that on S^d (stereographic transformation).

As

$$\frac{\partial}{\partial \tau} = \frac{1}{2}(1+u_+^2)\frac{\partial}{\partial u_+} + \frac{1}{2}(1+u_-^2)\frac{\partial}{\partial u_-} \,, \tag{22}$$

the generator H for global time translation on $\mathbb{R} \times S^{d-1}$ is identified with linear combination

$$H = \frac{1}{2}(P_0 + K_0) = J_{0,p+2}. \tag{23}$$

The existence of the generator H also guarantees that a correlation function of a CFT on $\mathbb{R}^{1,d-1}$ can be analytically extended to the entire Einstein static universe $\mathbb{R} \times S^{d-1}$.

3 Mapping global symmetry

A key necessary ingredient for the AdS/CFT correspondence to hold is that the global unbroken symmetries be identical. In the conformal phase, the continuous global symmetry of $\mathcal{N} = 4$ SYM is the superconformal group

$$SU(2, 2|4).$$

Its symmetries are generated by

1. Conformal symmetry $SU(2, 2) \sim SO(2, 4)$ for $\mathbb{R}^{1,3}$.
2. R-symmetry for $\mathcal{N} = 4$, $SU(4)_R \sim SO(6)_R$, generated by T^A, $A = 1, \cdots, 15$. And indeed $SU(4)_R$ is the automorphism group of the $\mathcal{N} = 4$ superconformal algebra. T^A are part of the algebra, and do not act just as external automorphisms.
3. Poincare supersymmetries generated by Q^a_α and their complex conjugates $\bar{Q}_{\dot{\alpha}a}$.
4. Conformal supersymmetries generated by $S_{a\alpha}$ and their complex conjugates $\bar{S}^a_{\dot{\alpha}}$.

The superconformal group $SU(2, 2|4)$ pertains the maximal bosonic subgroup

$$SU(2, 2) \times SU(4)_R \sim SO(2, 4) \times SO(6)_R,$$

and is readily recognized on the AdS side as the isometry group of the $AdS_5 \times S^5$ background.

The completion into the full supergroup 16 of the 32 Poincare supersymmetries are preserved by the array of N parallel D3- branes, and in the AdS limit, are supplemented by another 16 conformal supersymmetries. After including the fermionic generators required by supersymmetry, the full isometry supergroup of the $AdS_5 \times S^5$ background is $SU(2, 2|4)$. Thus the global symmetry $SU(2, 2|4)$ matches on both sides of the AdS/CFT correspondence.

It remains to show that the actual representations of the supergroup $SU(2, 2|4)$ also coincide on both sides. That is, we have to identify the contents of irreducible representations of $SU(2, 2|4)$ on the AdS side. We don't talk about this issue in this note.

4 Statement of AdS/CFT correspondence and correlation functions

The original strong form of AdS/CFT correspondence states the duality of the following theories

1. Type 2B string theory on $AdS_5 \times S^5$ where both AdS_5 and S^5 have the same radius L. The 5-form F_5^+ has integer flux $N = \int_{S^5} F_5^+$, and the string coupling is g_s.
2. $\mathcal{N} = 4$ super Yang-Mills theory in 4D, with gauge group $SU(N)$ and YM coupling g_{YM} in its (super)conformal phase.

Moreover, these two theories have the identifications of the parameters

$$4\pi g_s = g_{YM}^2, \qquad L^4 = 4\pi g_s N(\alpha')^2, \tag{24}$$

and the axion expectation value equals the SYM instanton angle

$$\langle C \rangle = \theta_I \ . \tag{25}$$

Remark 3. *The superconformal phase represents*

$$\langle X^i \rangle = 0 \qquad i = 1, \cdots, 6, \tag{26}$$

*where X^i are real scalars in the gauge multiplet, and are in the antisymmetric **6** of $SU(4)_R$. In this case, the gauge algebra $\mathcal{G}$ and the superconformal symmetry $SU(2,2|4)$ are unbroken.*

As we know, the YM coupling is related to the string coupling through

$$\tau \equiv \frac{4\pi i}{g_{YM}^2} + \frac{\theta}{2\pi} = \frac{i}{g_s} + \frac{\langle C \rangle}{2\pi} \ . \tag{27}$$

So changing the gauge coupling constant corresponds to changing the boundary value of the dilaton, which is achieved by a deformation by a marginal operator $\mathcal{O}(x)$ in SYM. To be more precise, the generic AdS/CFT correspondence could be expressed as the equation

$$\exp\Big(- S_{string}\big[\phi(x,z)|_{z=0} = \phi_0(x)\big]\Big) = \Big\langle \exp\Big(\int d^4x \phi_0(x)\mathcal{O}(x)\Big)\Big\rangle_{CFT} \ . \tag{28}$$

In this proposed equation, the LHS is the full partition function of string theory with boundary condition that $\phi = \phi_0$ on the boundary of AdS. While on the RHS is the generating function of the correlation functions in CFT. In the same way, theories which are related by the AdS/CFT correspondence are conjectured to be exactly equivalent, despite living in different numbers of dimensions.

Let's see the example of 2-point function, which encodes a straightforward application of the AdS/CFT correspondence.

4.1 SYM calculation

The composite operators in SYM is defined by

$$\mathcal{O}_\Delta(x) \equiv \frac{1}{n_\Delta} \mathrm{str} X^{i_1}(x) \cdots X^{i_k}(x), \tag{29}$$

where Δ is a collection of indices $\{i_1, \cdots, i_k\}$, and 'str' denotes the symmetrized trace over gauge algebra $\mathcal{G}$. The standard result of CFT evaluation give the 2-point function

$$\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\rangle = \frac{\delta_{\Delta_1,\Delta_2}}{(x_1 - x_2)^{2\Delta_1}} \ , \tag{30}$$

which reflects of conformal invariance.

4.2 AdS calculation

The propagators are considered in Euclidean AdS_{d+1}, a space denoted by H. (We are using the hyperbolic coordinates.) The action to linearized order reads

$$S = \int_H d^{d+1}z\sqrt{g}\left[\frac{1}{2}g_{\mu\nu}\partial_\mu\varphi_\Delta\partial_\nu\varphi_\Delta + \frac{1}{2}m^2\varphi_\Delta^2\right] - \int_{\partial H} d^d\boldsymbol{z}\bar{\varphi}_\Delta(\boldsymbol{z})\bar{J}(\boldsymbol{z}) \; , \tag{31}$$

where the boundary field $\bar{\varphi}_\Delta$ is related to the bulk field φ_Δ by

$$\bar{\varphi}_\Delta(\boldsymbol{z}) = \lim_{z_0\to 0} z_0^{\Delta-d}\varphi_\Delta(z_0,\boldsymbol{z}). \tag{32}$$

Here, $z_0 \to 0$ stands for approaching the boundary of H.

Upon classical calculation, the corresponding boundary-to-bulk propagator is the Poisson kernel

$$K_\Delta(z,\boldsymbol{x}) = C_\Delta\left(\frac{z_0}{z_0^2+(\boldsymbol{z}-\boldsymbol{x})^2}\right)^\Delta \; , \tag{33}$$

The bulk field generated in response to the boundary source $\bar{J}$ is given by

$$\varphi_\Delta(z) = \int_{\partial H} d^d\boldsymbol{x}K_\Delta(z,\boldsymbol{x})\bar{J}(\boldsymbol{x}) \; . \tag{34}$$

On the AdS side, the 2-point function to the lowest order is obtained by extracting the z_0^Δ behavior of the boundary to bulk propagator $K_\Delta(z,\boldsymbol{x})$, which gives

$$\lim_{z_0\to 0} z_0^{-\Delta}K_\Delta(z,\boldsymbol{x}) \sim \frac{1}{(\boldsymbol{z}-\boldsymbol{x})^{2\Delta}} \; , \tag{35}$$

in agreement with the behavior predicted by conformal invariance Eq.(30).