Topological Superconductor and Axion Field Theory

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ABSTRACT: This note is prepared for the journal club talk. Here we discuss the topological superconductor (TSC) model and its relation with axion field theory proposed by Qi, Witten and Zhang [1].

Contents

1 Generalities

Following previous discussions on the Witten type QFTs, here we provide an interlude focused on the Schwarz type QFTs, or its application in solid state system more precisely.
The most significant distinction (as I understand it) between phase orders governed by Landau's paradigm and topological orders is we usually cannot define a local order parameter, such as the magnetization M in magnet systems, to characterize a phase. Nonetheless, it seems that we can often formulate some effective theories probing physical responses to, say, electromagnetic perturbation, and the response function is characterized by some topological order parameter/quantum numbers. The punchline is these quantum numbers are often *quantized* by higher general principles.
For example, the long wavelength limit of integer quantum Hall effect is described by the effective action

$$I_{\text{eff}} = \frac{\nu}{4\pi} \int A \, \mathrm{d}A, \tag{1.1}$$

where A_μ is the U(1) gauge potential. The coefficient ν is quantized owing to gauge invariance (if the large gauge transformation is allowed.) Another similar example is 3 (spatial) dimensional time reversal invariant (TRI) topological insulators, whose topological response is given by the θ term

$$I_{\text{eff}} = \frac{\theta}{32\pi^2} \int \mathrm{d}^4x \, \epsilon^{\mu\nu\rho\lambda} F_{\mu\nu} F_{\rho\lambda}. \tag{1.2}$$

θ term also appears in high energy physics, where we consider axion electrodynamics on some occasions. However, as a topological *quantum number*, θ in this case is no longer dynamical. Quite the opposite, dictated by time-reversal symmetry, it is now quantized to be 0 or π (mod 2π).

While the previous 2 examples are about insulators, whose electromagnetic response can be examined without much ambiguity, it is natural to ask what is the analogous role for topological superconductors, whose U(1) symmetry are broken explicitly. The main theme of this talk (and the note) is to introduce a resolution given by Qi, Witten and Zhang [1]. They argue that the 3 dimensional topological superconductor has a similar effective topological response action,

$$\mathscr{L}_{\text{eff}} = \sum_i \frac{C_{1i}\theta_i}{64\pi^2}\epsilon^{\mu\nu\rho\lambda}F_{\mu\nu}F_{\rho\lambda}. \tag{1.3}$$

The index i is summed over all fermi surfaces and C_{1i} is the corresponding 1st Chern number. We notice that this Lagrangian actually resembles (1.2), and thus it is worthing clarifying at least 2 points.

1. θ_is in (1.3) are essentially dynamical variables describing fluctuating superconductor phases, to which the gauge fields A couples.

2. A is Higgsed in superconducting phase and therefore is a gapped degree of freedom in (1.3). While in (1.2), A remains gapless in long wavelength limit.

One may be curious that if the gauge fields A remains gapped in the bulk as we expect for superconductors, what is the fun of this game? In later sections we will give more discussions on what may be physically intriguing.

This note is organized as follows. We will first introduce/review the mean field theory of a 3+1 dimensional superconductor in terms of a BdG Hamiltonian, from which we extract the relevant effective degrees of freedom, the Weyl fermions defined Fermi surfaces. Next, we review some facts about the 4+1 dimensional TRI topological insulator and build the relation between 3+1 TRI TSC and 4+1 TRI TI. Using such 4+1 dimensional representation, we try to *derive* (1.3) as a dimension-reduced effective field theory. Consequently, we discuss what physical event we can study.

2 Weyl Fermion Representation from BdG Hamiltonian

In this section we first try to establish the *Weyl representation* for a generic BdG N-band Hamiltonian in 3+1 dimensions. A reference worth recommendation is the work by Qi, Hughes and Zhang [2], where in the Appendix, detailed derivation is given.

$$H = \frac{1}{2}\sum_{\mathbf{k}}(\psi^{\dagger}_{\mathbf{k}}, \psi^{T}_{-\mathbf{k}})\begin{pmatrix} h_{\mathbf{k}} & \Delta_{\mathbf{k}} \\ \Delta^{\dagger}_{\mathbf{k}} & -h^{T}_{-\mathbf{k}} \end{pmatrix}\begin{pmatrix} \psi_{\mathbf{k}} \\ \psi^{\dagger T}_{-\mathbf{k}} \end{pmatrix}, \tag{2.1}$$

where $\psi_{\mathbf{k}}$ represents a N-component column containing fermion operators from each band. h and Δ are both $N \times N$ matrices.
The time reversal symmetry T in this model is defined as

$$\mathsf{T}^{-1}\psi_{\mathbf{k}}\mathsf{T} = \mathcal{T}^{\dagger}\psi_{-\mathbf{k}}, \mathcal{T} = -\mathcal{T}^{T},\ \mathcal{T}^{\dagger}\mathcal{T} = \mathbb{I}. \tag{2.2}$$

The band Hamiltonian is TRI if

$$\mathcal{T}^{\dagger}h_{\mathbf{k}}\mathcal{T} = h^{T}_{-\mathbf{k}} \tag{2.3}$$

and the superconducting pairing has to satisfy

$$\Delta(\mathbf{k}) = -\Delta^{T}(-\mathbf{k}),\ (\mathcal{T}\Delta^{\dagger}(\mathbf{k}))^{\dagger} = \mathcal{T}\Delta^{\dagger}(\mathbf{k}). \tag{2.4}$$

At the level of band Hamiltonian, eigenstates to $h_{\mathbf{k}}$ can be characterized by 2 quantum numbers $(n, \mathbf{k})$ as $|n\mathbf{k}\rangle$. Depending on the value of chemical potential, Fermi surfaces can be determined. Making use of these notions we can define a Berry connection for each band

$$a^{n}_{i} = -\mathrm{i}\langle n\mathbf{k}|\partial_{k_i}|n\mathbf{k}\rangle \tag{2.5}$$

and an associating Chern-Number

$$C_{1n} = \frac{1}{2\pi}\int_{\Sigma_n} \mathrm{d}\Omega^{ij}(\partial_i a^{n}_{j} - \partial_j a^{n}_{i}), \tag{2.6}$$

where Σ_n is the Fermi Surface for nth band. Up to this point, we have not turned on the topological superconductor. A natural question is that how these insulator stories work as the superconductivity is turned one?
A helpful assumption is the weak-paring limit $|\Delta| \ll v_F K$, where K is the typical separation between adjacent Fermi surfaces. As argued in [2], in such limit, the formula (2.6) is still useful for characterizing topological phases.
The quick conclusion is the following. By the time reversal symmetry defined earlier in this paragraph, we may define a real matrix element

$$\delta_{n\mathbf{k}} = \langle n\mathbf{k}|\mathcal{T}\Delta^{\dagger}_{\mathbf{k}}|\mathbf{k}\rangle \tag{2.7}$$

and the total winding number is given by

$$N = \frac{1}{2}\sum_{n}\mathrm{sgn}(\delta_{n\mathbf{k}})C_{1n}. \tag{2.8}$$

Note that $\pm N$ are actually physically indistinguishable since the matrix $\mathcal{T}$ also has such ambiguity.

As an example illustrating the formalism above, let us consider the 2-band *minimal model*.

$$h_{\mathbf{k}} = \frac{\mathbf{k}^2}{2m} - \mu + \alpha\sigma \cdot \mathbf{k},\ \Delta_{\mathbf{k}} = \mathrm{i}\Delta_0\sigma_y\sigma \cdot \mathbf{k},\ \mathcal{T} = \mathrm{i}\sigma_y. \tag{2.9}$$

Since $\sigma \cdot \mathbf{k}$ has eigenvalues $\pm|\mathbf{k}|$, we know the energy eigenstates are also characterized by Weyl fermions with $\pm$ helicity. In particular, in this model we can find 2 Fermi sphere for $h_{\mathbf{k}}$, that $k_{F\pm} = \mp m\alpha + \sqrt{m^2\alpha^2 + 2m\mu^2}$, which have distinct helicities and thus have opposite Chern numbers. Moreover, $\mathcal{T}\Delta^\dagger = \Delta_0\sigma \cdot \mathbf{k}$ yields opposite signs on 2 Fermi sphere. This model thus has a non-trivial winding number sgn(Δ_0).

If we accept the idea that the topological phase is signaled by the winding number (2.8), it is critical that there are definite C_1 defined over Fermi surfaces. Besides, as we saw in the last example, the non-triviality is related to Weyl fermions as the low energy degrees of freedom near Fermi surfaces. We can imagine around Fermi surfaces, the effective Hamiltonian can be written as a sum of 2 Weyl fermions with $R(+)$ and $L(-)$ helicities respectively, while their superconducting pairings have opposite signs.

3 The 4+1 Dimensional Topological Insulator

By far we have yet introduced the electromagnetic field, since for a superconductor, U(1) symmetry is explicitly broken. However, we can talk about the electromagnetic response for a insulator. In particular, we are discussing the time-reversal invariant topological insulators in (4+1) dimensions [3].

Before jumping into higher dimensions, we recall the story in 2+1 dimensional band insulator without time reversal symmetry. We may also consider the Chern number for a filled band n

$$C_{1n} = \frac{1}{2\pi}\int_{\mathrm{BZ}} \mathrm{d}^2k\, \epsilon^{ij}\partial_i a_j^n. \tag{3.1}$$

We say this number characterize the electromagnetic response of the insulator in the sense that the ν in (1.1) is given by summing C_{1n} over all occupied bands.

Similar stories can be written in higher dimensions. In 4+1 dimensions, suppose there are N occupied bands. The U(N) Berry connection can be defined analogously

$$a_i^{nm} = -\mathrm{i}\langle n\mathbf{k}|\partial_{k_i}|m\mathbf{k}\rangle, \tag{3.2}$$

following which we have matrix-valued Berry curvature $f_{ij} = \partial_i a_j - \partial_j a_i + \mathrm{i}[a_i, a_j]$ and the second Chern number

$$C_2 = \frac{1}{32\pi^2}\int \mathrm{d}^4k\, \epsilon^{ijkl}\,\mathrm{Tr}[f_{ij}f_{kl}]. \tag{3.3}$$

This number characterizes the electromagnetic response as well in the sense that it is the coefficient of the effective Lagrangian

$$I_{\mathrm{CS}} = \frac{C_2}{24\pi^2}\int \mathrm{d}^5x\, \epsilon^{abcde} A_a \partial_b A_c \partial_d A_e. \tag{3.4}$$

Suppose the 4+1 dimensional manifold has boundaries. Looking at 1 of them, bulk-boundary correspondence indicates C_2 is the difference between left-handed and right-handed Weyl fermion numbers. If we consider a TRI TI with $C_2 = 1$ living in $\mathcal{M} = T^3 \times I$, where T^3 is the 3-torus and $I = [0, L_4]$. The bulk boundary correspondence then implies there is a copy of Weyl fermion ψ_L living on 1 boundary $x_4 = 0$, while on the other end $x_4 = L_4$ there is also a copy ψ_R with opposite helicity. Then by stacking superconductors with proper paring $|\Delta_i| e^{\mathrm{i}\theta_i} \psi^\dagger_{i-\mathbf{k}} \mathrm{i}\sigma_y \psi^\dagger_{i\mathbf{k}} +$ H.c. reproduces the low energy Lagrangian argued in the last section. This way we establish the mapping between the 3+1 dimensional TSC and the 4+1 dimensional TI.

4 The Effective Lagrangian

If we accept that the 3+1 dimensional TRI BdG Hamiltonian and the boundary states of 4+1 dimensional TRI topological insulator have the same relevant degrees of freedom in long wavelength limit, say, Weyl fermions, the next step we are taking is to introduce appropriate gap functions to the surface states of 4+1 dimensional TRI TI.
Suppose the 4th spatial dimension x_4 is of length L_4. Let us stack

$$\begin{aligned} H_{\mathrm{SC}} = &\int \mathrm{d}^3x\, \Delta_0\, e^{-\mathrm{i}\theta_L(x)} \psi^T(x_4 = 0) \mathcal{T} \psi(x_4 = 0) \\ &+ \int \mathrm{d}^3x\, \Delta_0\, e^{-\mathrm{i}\theta_R(x)} \psi^T(x_4 = L_4) \mathcal{T} \psi(x_4 = L_4) + \mathrm{H.C.} \end{aligned} \tag{4.1}$$

to the boundary of a 4+1 dimensional TRI TI. This boundary Hamiltonian, together with bulk terms, defines the theory. The U(1) global symmetry is one of the bulk TRI TI Hamiltonian. Let us turn on the background field A with $A_4 = 0$ and $A_\mu = A_\mu(t, x_1, x_2, x_3)$. Next we would like to integrate out fermion degrees of freedom ψ. To this end, we would like to separate them from the phase fluctuation θ. This can be done with the assist from the gauge transformation.

$$\psi \to \psi e^{\mathrm{i}\varphi(x)},\ A \to \tilde{A} = A + \partial\varphi \tag{4.2}$$

$$\theta_R \to \theta_R + 2\varphi(x_4 = L_4) \tag{4.3}$$

$$\theta_L \to \theta_L + 2\varphi(x_4 = 0). \tag{4.4}$$

By choosing

$$\varphi = -\frac{1}{2L_4}[\theta_L(L_4 - x_4) + \theta_R x_4], \tag{4.5}$$

we can eliminate the phase fluctuation on the boundaries and encode them into the gauge fields $\tilde{A}_4 = \partial_4\varphi = (\theta_R - \theta_L)/2L_4$. Integrating out the c fields then yields

$$I_{\mathrm{CS}} = \frac{1}{24\pi^2}\int \mathrm{d}^5x\, \epsilon^{abcde}\tilde{A}_a\partial_b\tilde{A}_c\partial_d\tilde{A}_e = \frac{1}{32\pi^2}\int \mathrm{d}^4x\, \epsilon^{\mu\nu\lambda\rho}\frac{\theta_L - \theta_R}{2}F_{\mu\nu}F_{\lambda\rho}. \tag{4.6}$$

This Chern-Simons term comes from fermion degrees of freedom. The full action incorporates contributions from the ordinary 3+1 dimensions such as the Higgsed Maxwell action and Higgs term. Besides, θ_L and θ_R are actually correlated in the original superconductor. A Josephson coupling is also allowed on this occasion. Consequently, the full action reads

$$\begin{aligned} I_{\mathrm{eff}} = \int \mathrm{d}^4x \bigg[&\frac{\theta_L - \theta_R}{64\pi^2}\epsilon^{\mu\nu\sigma\rho}F_{\mu\nu}F_{\sigma\rho} - \frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} \\ &+ \frac{1}{2}\rho_L(\partial_\mu\theta_L - 2A_\mu)^2 + \frac{1}{2}\rho_R(\partial_\mu\theta_R - 2A_\mu)^2 + J\cos(\theta_L - \theta_R)\bigg]. \end{aligned} \tag{4.7}$$

This action is TRI provided we define T as $\theta_i \to -\theta_i$, $A_0 \to A_0$ and $A^i \to -A^i$. This requirement constrains the form of Josephson coupling.
Maybe it's worth noting that the phase fluctuation here looks similar to one in the Landau-Ginzburg mean field description. Gauge symmetry of A is broken superficially, that is, it manifest in an usual way. In this case, it is realized in terms of a translation in θ_is. $\theta_i \to \theta_i + \alpha$ and $2A \to 2A + \partial\alpha$. Therefore, $\alpha =$ a constant is an U(1) global symmetry. Using Noether's recipe, it not difficult to derive a conserved current,

$$\left.\frac{\delta I_{\mathrm{eff}}}{\delta(\partial^\mu\alpha)}\right|_{\partial_\mu\alpha=0} = \sum_i \rho_i(\partial_\mu\theta_i - 2A_\mu). \tag{4.8}$$

Nonetheless, whether it is conserved needs the assistance of the equation of motion. Let us examine the equation of motions by varying I_{eff} with respect to A_μ. Up to boundary terms,

$$\delta_A I_{\mathrm{eff}} = \delta A_\rho\left[\frac{\epsilon^{\rho\sigma\mu\nu}}{8\pi^2}\partial_\sigma(\theta_L - \theta_R)\partial_\mu A_\nu - \frac{1}{e^2}\partial_\nu F^{\rho\nu} - 2\sum_i \rho_i(\partial^\rho\theta_i - 2A^\rho)\right]. \tag{4.9}$$

There is a term proportional to $\epsilon\partial F$, which vanishes due to Bianchi identity. Therefore,

$$\partial_\rho J^\rho = \partial_\rho(2\sum_i \rho_i(\partial^\rho\theta_i - 2A^\rho)) = \frac{1}{4\pi}\partial_\mu A_\nu\frac{\epsilon^{\mu\nu\rho\sigma}}{2\pi}\partial_\rho\partial_\sigma(\theta_L - \theta_R). \tag{4.10}$$

Due to the axion term, the Noether current is conserved only in the absence of vortex.
There is at least one more reason that we should be looking at the vortex lines in the effective theory. As we emphasized, the electromagnetic fields in the superconductor bulk is Meisner screened. Non-trivial gauge fields can exist only at vortices.
Therefore, conceptually we can identify the tensor $J_i^{\mu\nu} = \frac{1}{2\pi}\epsilon^{\mu\nu\sigma\tau}\partial_\sigma\partial_\tau\theta_i$ as kind of vortex current. For example, we can consider a chiral vortex line along z direction with non-vanishing Chern number C_L, while $C_R = 0$.

$$J_L^{zt} = -J_L^{tz} = \delta(x)\delta(y). \tag{4.11}$$

The non-conserving part of the current reads

$$\int \mathrm{d}^4x\,\partial_\rho J^\rho = \frac{1}{4\pi}\int \mathrm{d}^4x\,\delta(x)\delta(y)F_{zt} = \frac{1}{2}\int \mathrm{d}z\,\mathrm{d}t\,\frac{F_{zt}}{2\pi}. \tag{4.12}$$

Suppose the field strength satisfies conventional flux quantization. This anomaly equation is even more anomalous in the sense that the charge is fractionalized in units of $1/2$.
To provide this relation a physical interpretation, let us recall the axial anomaly in Schwinger model, in which the anomalous part of a Dirac fermion is

$$\partial_\mu J_5^\mu = \frac{1}{2\pi}\epsilon^{\mu\nu}F_{\mu\nu},\ \mu,\nu\in\{0,1\}. \tag{4.13}$$

Together with vectorial current conservation, the anomalous part for a Weyl fermion is $1/2$ of (4.13). (4.12) is yet a half of the Weyl fermion anomaly. Therefore, a possible interpretation is that under a unit flux of F_{zt}, anomaly is lead by a Weyl-Majorana fermion propagating in the 1+1 dimensional vortex line.
We may try to consider a yet illuminating example by considering an instanton event. Again we assume periodic boundary conditions in 3 spatial directions x, y and z. Then we consider a set of vortex lines including chiral ones and ordinary ones.
Since the manifold is now compact, fluxes contributed from each 2-hypersurface should be quantized in units of 2π. This sets a constraint on the number of chiral vortices. This can be seen from (4.9). Far from the vortex centers, field strength is Meisner screened. The equation of motion reduces to

$$\rho_L(\partial_\mu\theta_L - 2A_\mu) + \rho_R(\partial_\mu\theta_R - 2A_\mu) = 0. \tag{4.14}$$

A line integral enclosing a chiral vortex θ_i hence yields the flux threading through the plane

$$\oint \mathrm{d}\boldsymbol{\ell}\cdot\mathbf{A} = \frac{\rho_i}{\rho_L+\rho_R}\pi. \tag{4.15}$$

As a consequence, the number of chiral vortices must be a multiple of 4. Half of them are right-handed and the other half are left-handed. Similarly, ordinary vortices come in pairs. Consequently, let us consider a minimal configuration. 2 right-handed and 2 left-handed vortices go in y direction. 2 ordinary vortex lines go in z direction, while they are also moving in x direction at constant speed v. In addition, to simplify the argument, we assume $\rho_L = \rho_R$ so that we can write, for each chiral vortex,

$$F^{\text{cv}}_{zx} = \frac{\pi}{2}\delta(x - x_0)\delta(z - z_0), \tag{4.16}$$

and for each ordinary vortex,

$$F^{\text{ov}}_{xy} = \pi\delta(x - vt)\delta(y - y_0) \tag{4.17}$$

$$F^{\text{ov}}_{ty} = -v\pi\delta(x - vt)\delta(y - y_0). \tag{4.18}$$

Plugging them into the axion term yields the mutual interaction

$$2\epsilon^{\mu\nu\lambda\rho}F^{\text{cv}}_{\mu\nu}F^{\text{ov}}_{\lambda\rho} = -8F^{\text{cv}}_{zx}F^{\text{ov}}_{ty} = 4\pi^2\delta(t - x_0/v)\delta(x - x_0)\delta(y - y_0)\delta(z - z_0). \tag{4.19}$$

There are in total 8 such terms. Thus,

$$\int \mathrm{d}^4x\, \frac{\theta_L - \theta_R}{64\pi^2}\epsilon^{\mu\nu\lambda\rho}F_{\mu\nu}F_{\lambda\rho} = \frac{1}{16}\sum_{n=1}^{8}[\theta_L(x_n) - \theta_R(x_n)], \tag{4.20}$$

where x_ns are spacetime coordinates where vortex lines intersect. After 2 ordinary vortex lines pass all 4 chiral vortex lines. Periodicity indicates that the system goes back to its starting state up to a topological phase (4.20). Imagine a global gauge transformation where $\theta_R \to \theta_R + 2\pi$ with θ_L fixed. (4.20) give π and thus the total partition function gets a -1. We may understand the minus sign in the following way. $\theta_R \to 2\pi$ correspond to a phase transformation for the chiral fermion as well $\psi_R \to e^{\mathrm{i}\pi}\psi_R = -\psi_R$. Therefore, the whole many body ground state obtains $(-1)^{N_R}$ under such global transformation. The -1 here indicates after the instanton event, the final ground state now owns different fermion parity $(-1)^{N'_R}$. Such a change in fermion number can be interpreted as residual chiral anomaly of 3+1 dimensional chiral anomaly. Going to Peskin and Schroeder we can quote the anomaly equation

$$\delta Q_L = -\delta Q_R = \frac{1}{32\pi^2}\int \mathrm{d}^4x\, \epsilon^{\mu\nu\lambda\rho}F_{\mu\nu}F_{\lambda\rho} = C_2. \tag{4.21}$$

The instanton event we just considered makes the right-hand side 1.
It is dubbed *residual* since in the context of superconductor we do not anticipate fermion number conservation. However, this anomaly stays *anomalous* because it violates fermion number parity conservation with the vortex crossing process we considered above.

5 Summary and Comment

In this note we have shown the effective topological response of a TRI TSC can be described in a similar way to a TRI TI by θ terms. Taking all terms allowed, we have a general effective description. We also discuss what can be taught by the effective action with an example of instanton event. We show that resulting anomaly equation indicates violation of fermion parity conservation.

This paper is one of the papers applying Chern Simons theories in various dimensions to describing long-wavelength effective theories in solid states, and the author hopes it may serve as a fine introduction to the topological field theory of Schwarz type.

As a final remark, due to the similarity between the effective actions of TRI TI and TRI TC in 3+1 dimensions, there is work arguing they are actually Maxwell dual to each other in some limits. Motivated by such correspondence, a similar duality is proposed in 2+1 dimensions.

An insightful comment given by Paul Wiegmann is that similar physics has been thoroughly studied in the community of ^{3}He regardless of theoretical or experimental aspect. A comprehensive text is *The Superfluid Phases of Helium 3* by D. Vollhardt and P. Wölfle. There is also a recent review [4] that readers can reference.

References

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