Precision Holography with Supersymmetric Wilson Loops

Vimal Rathee
Graduate Student, University of Michigan

Great Lakes Strings 2018, UChicago

April 14, 2018

1802.03016, 1802.06789 J. Aguilera Damia, A. Faraggi, L. Pando Zayas, V. Rathee, G. A Silva
Motivation

- AdS/CFT beyond the leading order:

\[ Z_{Field Theory}(\phi_0) = Z_{String} \approx \exp(-S(\phi \to \phi_0)_{Grav}). \]

- Localization: A plethora of exact results for supersymmetric field theories.

- What can we learn about string perturbation theory given the “experimental data” provided by localization?
Wilson Loops (WL) are an important class of gauge invariant non-local operators.

$$W_R (C) = tr_R \mathcal{P} \exp \oint_C (A_\mu \, dx^\mu)$$

The expectation value measures the effective action of an external particle; order parameter for confinement, $V_{q\bar{q}} \propto r$. 

[arxiv:1412.8008]
- Wilson loop in Gauge/Gravity Correspondence
- The contour becomes a surface in higher dimensions.
- Expectation value:

\[ \langle W(C) \rangle = Z_{\text{string}} (\partial \Sigma = C) \]

- Right regime

\[ Z_{\text{string}} (\partial \Sigma = C) = e^{-S(C)} \]
Matrix Model: Half BPS Wilson Loop in $\mathcal{N} = 4$ SYM

- The Matrix Model computation gives the exact answer, for any $N$ and $\lambda$, in terms of Laguerre polynomial,

$$\langle W^{\Box} \rangle_{\text{circle}} = \frac{1}{N} L_{N-1}^{1} \left( -\frac{\lambda}{4N} \right) e^{\lambda/8N}$$

$$\approx \frac{2}{\sqrt{\lambda}} I_{1}(\sqrt{\lambda}) + \frac{\lambda}{38N^{2}} I_{2}(\sqrt{\lambda}) + \frac{\lambda^{2}}{1280N^{4}} I_{4}(\sqrt{\lambda}) + \ldots$$

$$\approx \exp \left( \sqrt{\lambda} - \frac{3}{4} \ln \lambda - \frac{1}{2} \ln \frac{\pi}{2} + \ldots \right)$$
Gravity Side: Beyond the leading order

- Forste-Ghoshal-Theisen 9903042, Drukker-Gross-Tseytlin 0001204, Kruczenski-Tirziu 0803.0315, Buchbinder-Tseytlin 1404.4952.

\[ \langle W \rangle = \exp(-\Gamma), \quad \Gamma = \Gamma_0 + \Gamma_1, \]
\[ \Gamma_1 = \frac{1}{2} \ln \frac{[\det(-\nabla^2 + 2)]^3[\det(-\nabla^2)]^5}{[\det(-\nabla^2 + \frac{1}{4} R^{(2)} + 1)]^8} \]

- Five massless modes \((S^5)\); three massive modes \(AdS_2 \subset AdS_5\).

\[ \langle W \Box \rangle_{\text{circle}} = \exp \left( \sqrt{\lambda} - \frac{1}{2} \ln(2\pi) + \ldots \right) \]
WL beyond the leading order: Problem/Opportunity

\[ \langle W_{\Box} \rangle_{\text{circle}} = \exp \left( \sqrt{\lambda - \frac{3}{4} \ln \lambda} - \frac{1}{2} \ln \frac{\pi}{2} + \ldots \right) \]

\[ \langle W_{\Box} \rangle_{\text{circle}} = \exp \left( \sqrt{\lambda - \frac{1}{2} \ln (2\pi)} + \ldots \right) \]

- Missing the \( \ln(\lambda) \) term on the gravity side (zero modes, etc.).
- Numerical discrepancy is not an error: Drukker-Gross-Tseytlin 0001204, Kruczenski-Tirziu 0803.0315, Buchbinder-Tseytlin 1404.4952.
Toward a Precision Computation

- The AdS/CFT formula:

\[< W(C) >_{CFT} = < W(\Sigma \rightarrow C) >_{String}\]

- What we are doing:

\[Z_{string} = \exp(-S_{classical}) \times Z_{1-loop} \times \text{(Topological Sector)}\]

- We are missing aspects of string perturbation theory: Ghost zero modes, etc.

- Compare configurations with the same world sheet topology!

- The $1/4$ BPS Wilson loop beyond the leading order.
Holographic 1/4 BPS WL

- Classical solution – Ansatz:
  \[ \psi = \tau, \quad \sinh \rho = \frac{1}{\sinh \sigma}, \]
  \[ u = 0 \]
  \[ \phi = \tau, \quad \sin \theta = \frac{1}{\cosh (\sigma_0 + \sigma)}, \]

- Classical solution – World-sheet metric:
  \[ ds^2 = \left( \frac{1}{\sinh^2 \sigma} + \frac{1}{\cosh^2 (\sigma_0 - \sigma)} \right) (d\tau^2 + d\sigma^2). \]
Fluctuations

Forini-Giangreco-Griguolo-Seminaro-Vescovi [1512.00841], Faraggi-Pando Zayas-Silva-Trancanelli [1601.04708]

- The quadratic action for the bosonic fluctuations is

\[
S^{2,3,4} = \frac{\sqrt{\lambda}}{4\pi} \int d\tau d\sigma \sqrt{g} \left( g^{ab} \partial_a \chi \partial_b \chi + \frac{2}{\sqrt{g}} \chi^2 \right),
\]

\[
S^{5,6} = \frac{\sqrt{\lambda}}{2\pi} \int d\tau d\sigma \sqrt{g} \left( g^{ab} D_a \chi (D_b \chi)^\dagger - \frac{2m^2(\sigma_0)}{\sqrt{g}} |\chi|^2 \right),
\]

\[
S^{7,8,9} = \frac{\sqrt{\lambda}}{4\pi} \int d\tau d\sigma \sqrt{g} \left( g^{ab} \partial_a \chi \partial_b \chi - \frac{2 \sin^2 \theta}{\sqrt{g}} \chi^2 \right).
\]
One-loop effective action

\[ e^{-\Gamma_{\text{effective}}^{1-\text{loop}}} = \frac{(\text{Det } \mathcal{O}^+)^{\frac{4}{2}} (\text{Det } \mathcal{O}^-)^{\frac{4}{2}}}{(\text{Det } \mathcal{O}^{2,3,4})^{\frac{3}{2}} (\text{Det } \mathcal{O}^{5,6})^{\frac{2}{2}} (\text{Det } \mathcal{O}^{7,8,9})^{\frac{3}{2}}} , \]

\[ \ln (\text{Det } \mathcal{O}) = \sum_{E} \ln (\text{Det } \mathcal{O}_E) , \]

\( \mathcal{O}_E \) is the corresponding one-dimensional operator acting on a specific Fourier mode.
Ratio of 1-loop effective action

- For 1/4 BPS dependence on the value of $\sigma_0$ that characterizes the classical string solution. The 1/2 BPS is $\sigma_0 \to \infty$.

$$
\Omega_{E}^{2,3,4}(\sigma_0) = \ln \left[ \frac{\text{Det } O_{E}^{2,3,4}(\sigma_0)}{\text{Det } O_{E}^{2,3,4}(\infty)} \right],
$$

$$
\Omega_{E}^{5,6}(\sigma_0) = \ln \left[ \frac{\text{Det } O_{E}^{5,6}(\sigma_0)}{\text{Det } O_{E}^{5,6}(\infty)} \right],
$$

$$
\Omega_{E}^{7,8,9}(\sigma_0) = \ln \left[ \frac{\text{Det } O_{E}^{7,8,9}(\sigma_0)}{\text{Det } O_{E}^{7,8,9}(\infty)} \right],
$$

$$
\Omega_{E}^{\alpha}(\sigma_0) = \ln \left[ \frac{\text{Det } O_{E}^{\alpha}(\sigma_0)}{\text{Det } O_{E}^{\alpha}(\infty)} \right]
$$

- Each ratio is to be computed using the Gelfand-Yaglom (Coleman) method.
\[ \Delta \Gamma_{\text{1-loop effective}}^{1-\text{loop}}(\sigma_0) = \frac{1}{2} \sum_{E \in \mathbb{Z}} \left( 3 \Omega_{E}^{2,3,4}(\sigma_0) + 2 \Omega_{E}^{5,6}(\sigma_0) + 3 \Omega_{E}^{7,8,9}(\sigma_0) \right) \]
\[ - \frac{4}{2} \sum_{E \in \mathbb{Z} + \frac{1}{2}} \left( \Omega_{E}^{+}(\sigma_0) + \Omega_{E}^{-}(\sigma_0) \right) . \]

Result

\[ \Delta \Gamma_{\text{1-loop effective}}^{1-\text{loop}} = \frac{3}{2} \ln \tanh \sigma_0 - \ln \sqrt{\frac{1 + \tanh \sigma_0}{2}} \]
\[ = \frac{3}{2} \ln \cos \theta_0 - \ln \cos \frac{\theta_0}{2} , \]

The first term gives the predicted result from the gauge theory. It comes from the \( E = 0 \) mode of the \( \Omega_{E}^{7,8,9} \) determinant.
Revision 1

- Forini-Tseytlin-Vescovi [1702.02164] have proposed a perturbative heat kernel computation that matches the field theory answer.
- Hard to compute heat kernels in non-homogeneous spaces. Expand the heat kernel around $\theta_0 = 0$.

$$
\begin{align*}
g_{ij} & = g_{ij}^{AdS_2} + \theta_0^2 \tilde{g}_{ij} + O(\theta_0^4), \\
\mathcal{O} & = \mathcal{O}^{AdS_2} + \theta_0^2 \tilde{\mathcal{O}} + O(\theta_0^4), \\
K_{\mathcal{O}}(x, x'; t) & = K_{\mathcal{O}}^{AdS_2}(x, x'; t) + \theta_0^2 \tilde{K}_{\mathcal{O}}(x, x'; t) + O(\theta_0^4),
\end{align*}
$$

- What does it teach us? The Gelfand-Yaglom method might be imposing very strict conditions.
Revision 2

- Cagnazzo-Medina Rincon-Zarembo [1712.07730]
- Mapping problem to cylinder

![Diagram of a mapping problem to a cylinder]

- Diffeomorphism-invariant Regulator

\[ R \equiv R_{\text{inv}} - \ln \cos \frac{\theta_0}{2} \]

- Well defined problem on disk.
Zeta-function Regularization

Motivation from Dunne-Kirsten [0607066], Non-perturbative result for one-loop corrections.

\[
\ln \frac{\text{det}O}{\text{det}O_{\text{free}}} = -\ln (\mu^2) \hat{\zeta}_O(0) - \hat{\zeta}'_O(0), \quad \hat{\zeta}_O(s) \equiv \zeta_O(s) - \zeta_{\text{free}}(s),
\]

where \( \mu \) is renormalization parameter.

- **Induced Geometry**
  \[ ds^2_M = M \, ds^2_{\text{AdS}_2} = M(\rho) \left( d\rho^2 + \sinh^2 \rho \, d\tau^2 \right) \]

- **Fluctuation operators**

  \[
  O_M = M^{-1} \, O_{\text{AdS}_2}, \quad \text{(bosons)} \\
  O_M = M^{-1/2} \, O_{\text{AdS}_2}, \quad \text{(fermions)}
  \]
\[ \mathcal{O}_{AdS_2}^B = -g^{\mu\nu} D_\mu D_\nu + m^2 + V, \quad D_\mu = \nabla_\mu - \iota q A_\mu. \]

\[
\ln \frac{\det \mathcal{O}_{AdS_2}}{\det \mathcal{O}_{AdS_2}^{\text{free}}} = \ln \frac{\det \mathcal{O}_0}{\det \mathcal{O}_0^{\text{free}}} + \sum_{l=1}^{\infty} \left( \ln \frac{\det \mathcal{O}_l}{\det \mathcal{O}_l^{\text{free}}} + \ln \frac{\det \mathcal{O}_{-l}}{\det \mathcal{O}_{-l}^{\text{free}}} + \frac{2}{l} \hat{\zeta}_\Phi(0) \right) \\
\quad - 2 \left( \gamma + \ln \frac{\mu}{2} \right) \hat{\zeta}_\Phi(0) + \int_0^\infty d\rho \, \sinh \rho \ln(\sinh \rho) V \\
\quad - q^2 \int_0^\infty d\rho \, \frac{A^2}{\sinh \rho},
\]

\[ \hat{\zeta}_\Phi(0) = -\frac{1}{2} \int_0^\infty d\rho \, \sinh \rho \, V, \]

Anomaly :

\[
\ln \frac{\det \mathcal{O}_M}{\det \mathcal{O}_M^{\text{free}}} = \ln \frac{\det \mathcal{O}_{AdS_2}}{\det \mathcal{O}_{AdS_2}^{\text{free}}} + \frac{1}{4\pi} \int d^2\sigma \sqrt{g} \ln M \left[ m^2 + V - \frac{R}{6} + \frac{1}{12} \nabla^2 \ln M \right].
\]
Conclusions and Open Directions

- Precision holography is a painstaking job and we are learning how to perform holographic computations.
- There are various venues in $\mathcal{N} = 4$ and ABJM.

Hints of string localization?
Explicit hints of a localization structure? What are these modes the localization locus of?
Is there a similar bulk localization mechanism for D-branes (corresponding to Wilson loops in higher dimensional representations)?

Similar techniques have been applied by Ashoke Sen and collaborators to asymptotically flat black holes. We should apply to the entropy of asymptotically AdS black holes.
Conclusions and Open Directions

- Precision holography is a painstaking job and we are learning how to perform holographic computations.
- There are various venues in $\mathcal{N} = 4$ and ABJM.
- Hints of string localization?
- Explicit hints of a localization structure? What are these modes the localization locus of?
- Is there a similar bulk localization mechanism for D-branes (corresponding to Wilson loops in higher dimensional representations)?
Conclusions and Open Directions

- Precision holography is a painstaking job and we are learning how to perform holographic computations.
- There are various venues in $\mathcal{N} = 4$ and ABJM.
- **Hints of string localization?**
- Explicit hints of a localization structure? What are these modes the localization locus of?
- Is there a similar bulk localization mechanism for D-branes (corresponding to Wilson loops in higher dimensional representations)?
- Similar techniques have been applied by Ashoke Sen and collaborators to asymptotically flat black holes. We should apply to the entropy of asymptotically AdS black holes.
Thanks!