Thermalization and Random Matrices

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Thermalization of Quantum Systems

How isolated quantum systems thermalize? Systems without additional symmetries – Eigenstate Thermalization Hypothesis

- Individual energy eigenstate is “thermal”

\[
\langle E | A | E \rangle \simeq \text{Tr}(\rho_{\text{mic}} A) \simeq \text{Tr}(e^{-\beta H} A) / \text{Tr}(e^{-\beta H})
\]

- “Eigenstate Ensemble” explains eventual thermalization

\[
\lim_{t \to \infty} \langle \Psi(t) | A | \Psi(t) \rangle = \sum_i |C_i|^2 \langle E_i | A | E_i \rangle + \\
\lim_{t \to \infty} \sum_{i \neq j} C_i^* C_j \langle E_i | A | E_j \rangle e^{-i(E_i-E_j)t} \simeq A^{\text{th}} + O(1/L)
\]
Motivation

- Thermalization after a quantum quench
  AD and Smolkin, arXiv:1709.08654

- ETH in CFT, chaotic CFTs, GGE for 2d CFTs

- Collapse of Black Holes as thermalization

- Thermalization in SYK, connection to random matrices and quantum chaos
Eigenstate Thermalization Hypothesis

ETH ansatz

\[ \langle E_i | A | E_j \rangle = A^{\text{eth}}(E) \delta_{ij} + \Omega^{-1/2} f(E, \omega) r_{ij} \]

\[ E = (E_i + E_j)/2, \quad \omega = E_i - E_j \]

\[ A^{\text{eth}}, f \] depend on energy density \( E/V \)

Deutsch’91  Srednicki’94; 99 Rigol, Dunjko, Olshanii’08

Meaning of form-factor \( f(\omega) \):

\[ \langle A(t) A(0) \rangle_\beta = \int d\omega f^2(E, \omega) e^{-i\omega t} \]
Chaoticity, ETH and Random Matrices

- Chaotic behavior: Hamiltonian = Random Matrix (WD distribution of energy levels)

- ETH $\approx$ Eigenstates are random vectors

- “random” behavior of $r_{ij}$, i.e. $A_{ij}$ with $i \neq j$ (empirical evidence)

- universal “ergodic” behavior of observables $\langle \Psi | A(t) | \Psi \rangle$ for large $t$ (after thermalization)? “structureless” or Haar-invariant $A_{ij}$

D’Alessio, Kafri, Polkovnikov, Rigol’15
Cotler et al., ’16, ’17
ETH reduces to RMT?

- For small $\omega \leq \tau^{-1}$, $f(\omega)$ is constant and $r_{nm}$ is GOE

$$\langle E_i | A | E_j \rangle = A^{\text{eth}} \delta_{nm} + \Omega^{-1/2} f(\omega) r_{ij}$$

D’Alessio, Kafri, Polkovnikov, Rigol’15

- Gaussian distribution of $r_{ii}$ and $r_{ij}$

  Beugeling, Moessner, Haque’14, . . .

- Ratio $\langle r_{ii}^2 \rangle = 2 \langle r_{ij}^2 \rangle$

  AD and Liu, arxiv:1702.07722, Mondaini, Rigol’17

What is the timescale when ETH reduces to RMT?

Is it $\Delta E_{RMT} = \tau^{-1}$-inverse Thouless time
Diffusive system thermalizes within Thouless time $\tau \sim L^2$ necessary for the slowest diffusive modes to propagate across the system. After time $t \sim \tau$ the system is fully ergodic (and ETH reduces to RMT).
The key idea: dynamics of “slow states” constraints
\[ \Delta E_{\text{RMT}} \]
Classical diffusion in 1D

\[
\frac{\partial \rho}{\partial t} = D \frac{\partial^2 \rho}{\partial x^2}
\]

\[
\rho(t, x) = \sum_n c_n \cos \left( \frac{\pi n x}{L} \right) e^{-tD(\pi n)^2/L^2}
\]
Quasi-classical slow states

- there are states $\Psi$ such that $\langle \Psi | \delta A(t) | \Psi \rangle$ remains of order one long time $t \sim \tau$, where $\delta A = A - A^{\text{eth}}$

$$\langle \Psi | \delta A(t) | \Psi \rangle \sim e^{-t/\tau}$$

- let's consider the deviation $\delta A(t)$ averaged over time $T$

$$\int dt \langle \Psi | \delta A(t) | \Psi \rangle \frac{\sin(\pi t/T)}{\pi t} \approx \frac{1}{T} \int_0^T dt \langle \Psi | \delta A(t) | \Psi \rangle \sim \frac{\tau}{T}$$

- for any local system $\tau \geq L$, for a diffusive system $\tau \sim L^2$, for non-local SYK system $\tau \sim ???$
From time domain to energy domain and back

Idea: to go from energy domain to time domain

\[
\int dt \frac{\sin(\pi t/T)}{t\pi} \langle \Psi(t) | \delta A | \Psi(t) \rangle = \langle \Psi(0) | \delta A_T | \Psi(0) \rangle
\]

\[
(\delta A_T)_{ij} = \begin{cases} 
\delta A_{ij} : |\omega| \leq 1/T \\
0 : |\omega| > 1/T 
\end{cases}
\]

\[
\delta A_T = \begin{bmatrix}
* & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & * \\
\end{bmatrix}
\]

\[\delta A_T\] is a matrix with band structure: within the diagonal band it coincides with \(A_{ij}\) with the diagonal \(A^\text{eth}_{ij}\) part removed, and zero outside
Upper bound on $\lambda$ of band matrix

- Value of $\langle \Psi(0)|\delta A_T|\Psi(0)\rangle$ is bounded by largest eigenvalue $\lambda(\delta A_T)$ of $\delta A_T$

- Let's introduce $\lambda(\Delta E, E)$ for the largest (by absolute value) eigenvalue of the sub-matrix centered at $E$ and of size $2\Delta E$

$$\lambda(\delta A_T) \leq 2\lambda(E', 2/T) + \lambda(E'', 1/T)$$
Band Random Matrices

- full band matrix $\delta A_T$ may not be random even when $1/T$ is very small (band is narrow) - because of possible correlations along the diagonal

- by assumption, when $2/T \leq \Delta E_{RMT}$, quadratic sub-matrices of size $\Delta E \leq 2/T$ or smaller are random

- assuming fluctuations $r_{ij}$ are independent

$$\lambda^2(\Delta E) \leq 8 \int_0^{1/T} d\omega |f(\omega)|^2$$

AD and Liu arxiv:1702.07722

this bound is uniform for all sizes $\Delta E \geq 1/T$ and only depends on the band-width $1/T$
Upper bound on $\Delta E_{\text{RMT}}$ from slow states

- for sufficiently large $T$, such that $T \Delta E_{\text{RMT}} \geq 2$

$$\max_{\Psi} \left| \int dt \frac{\sin(\pi t/T)}{t \pi} \langle \Psi(t) | \delta A | \Psi(t) \rangle \right|^2 \leq \int dt \frac{\sin(\pi t/T)}{t \pi} \langle A(t) A(0) \rangle_{\beta}$$

- 2pt function approaches $L$-independent asymptotic form in the thermodynamic limit $\langle A(t) A(0) \rangle_{\beta} \sim (t_D/t)^{\alpha}$

for 1D diffusive system $\alpha = 1/2$; when the system is finite

$$\int dt \frac{\sin(\pi t/T)}{t \pi} \langle A(t) A(0) \rangle_{\beta} \sim \left\{ \begin{array}{ll} \sqrt{t_D/T} & T \leq \tau \\ \sqrt{t_D \tau}/T & T \geq \tau \end{array} \right.$$  

- taking $\Psi$ to be a slow diffusive mode $\langle \Psi | \delta A(t) | \Psi \rangle \sim e^{-t/\tau}$

$$\left( \frac{\tau}{T} \right)^2 \leq \frac{\sqrt{t_D \tau}}{T} \Rightarrow T \geq L^3$$
Conclusions

- The “Random Matrix” time-scale $\Delta E_{\text{RMT}}^{-1}$, when ETH reduces to Random Matrix Theory, is parametrically longer than the Thouless time.

- What are the observational signatures of $\Delta E_{\text{RMT}}^{-1}$? Is there “ergodicity” and “universality” of $\langle \Psi | A(t) | \Psi \rangle$, or in the end of the story $\Delta E_{\text{RMT}} = 0$?

  (Hamiltonian = Random Matrix; observable is never random, rather some matrix written in random basis)

- Given that $A_{ij}$ is not “structureless” at Thouless energy scale, what happens at the “end of thermalization” $t \sim \tau$?
What’s the Big Picture?

- A new picture of thermalization with the new “Random Matrix” time-scale \( \Delta E_{\text{RM}}^{-1} \)

- The take home point: random matrices are not adequate to describe slow thermalization dynamics. What is the relation between Thouless time defined through spectrum properties and Thouless time defined as thermalization time for many-body systems?

- What are the relevant energy/timescales for the non-local SYK model and what is their bulk interpretation?