The topologically twisted index on $S^2 \times T^n$ and black hole entropy

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JTL, L. A. Pando Zayas, V. Rathee and W. Zhao

Junho Hong and JTL, arXiv:1804.04592
JTL, L. A. Pando Zayas and S. Zhou, in progress
Supersymmetric partition functions from localization

- Localization is a very powerful tool for computing supersymmetric partition functions and observables
  - $S^n$ partition functions, Wilson loop observables
  - $S^n \times S^1$ partition functions and supersymmetric indices

- Generically, the partition function takes the form

$$Z_{\text{susy}} = \int d\Phi Z_{\text{classical}} Z_{\text{1-loop}} Z_{\text{non-perturbative}}$$
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$$Z_{\text{susy}} = \int d\Phi \, Z_{\text{classical}} \, Z_{1\text{-loop}} \, Z_{\text{non-perturbative}}$$

- We explore the connection between the topologically twisted index on $S^2 \times S^1$ and AdS$_4$ black hole entropy and $S^2 \times T^2$ and AdS$_5$ black string microstates

Magnetically charged AdS solutions

- AdS/CFT allows us to compare observables on both sides of the duality

- Global AdS $\leftrightarrow$ partition function on $S^n$
- Black holes in AdS $\leftrightarrow$ partition function on $S^{n-1} \times S^1$
- Black strings in AdS $\leftrightarrow$ partition function on $S^{n-2} \times T^2$
Magnetically charged AdS solutions

- AdS/CFT allows us to compare observables on both sides of the duality

  | Global AdS | ↔ | partition function on $S^n$ |
  | Black holes in AdS | ↔ | partition function on $S^{n-1} \times S^1$ |
  | Black strings in AdS | ↔ | partition function on $S^{n-2} \times T^2$ |

- Consider a magnetic BPS black hole with spherical horizon

  | boundary | near horizon |
  | AdS | AdS$_4$ | $\rightarrow$ | AdS$_2 \times S^2$ |
  | $\Downarrow$ | $\Downarrow$ |
  | CFT | $S^2 \times S^1$ | $\rightarrow$ | $S^1$ |
The topologically twisted index on $S^2 \times S^1$

- The topologically twisted index was introduced by F. Benini and A. Zaffaroni, arXiv:1504.03698
- Take a magnetic BPS black hole in AdS$_4$
  What do we do on the field theory side?
  - Background $R$ symmetry flux on $S^2$
  - This cancels the curvature of $S^2 \Rightarrow$ partial topological twist
  - The index may be computed using localization
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- This topologically twisted index is conjectured to count the black hole microstates [Benini, Hristov, Zaffaroni]
  - Many general features are now known
  - Extended to dyonic black holes, black holes with hyperbolic horizons, magnetic black strings, ...
Building blocks of the $S^2 \times S^1$ index

- Consider three-dimensional $\mathcal{N} = 2$ Chern-Simons-matter theories on $S^2 \times S^1$

- The index receives contributions from:
  - Vector multiplets:
    \[
    Z_{\text{vector}} = \prod_i \frac{dx_i}{2\pi i x_i} x_i^{k_i} \prod_{\alpha \in G} (1 - x^\alpha)
    \]
  - Chiral multiplets:
    \[
    Z_{\text{chiral}} = \prod_{\mu \in R} \left( \frac{x^{\mu/2} y^{\mu_f/2}}{1 - x^{\mu} y^{\mu_f}} \right)^{\mu(m) + \mu_f(n) - q + 1}
    \]

- These elements can be combined to construct the index for various models
Counting black hole microstates

Given a magnetically charged AdS black hole, we can construct the topologically twisted index in the field theory dual and evaluate it in the large-$N$ limit.
Counting black hole microstates

- Given a magnetically charged AdS black hole, we can construct the topologically twisted index in the field theory dual and evaluate it in the large-$N$ limit.

- We consider the following examples:
  1. Magnetic black holes in M-theory on $\text{AdS}_4 \times S^7/\mathbb{Z}_k$
     Dual to ABJM theory
  2. Magnetic black holes in massive IIA on $\text{AdS}_4 \times S^6$
     Dual to $\mathcal{N} = 2$ Chern-Simons-matter theory
  3. Magnetic black strings in IIB on $\text{AdS}_5 \times S^5$
     Dual to $\mathcal{N} = 4$ super-Yang-Mills
M-theory on $\text{AdS}_4 \times S^7/Z_k$

- The field theory dual is ABJM theory
  Chern-Simons-matter with $U(N)_k \times U(N)_{-k}$ gauge groups and bi-fundamental matter $A_i, B_j$
- The topologically twisted index is given by

$$
Z(y_a, n_a) = \frac{1}{(N!)^2} \sum_{m, \tilde{m}} \int \prod_i \frac{dx_i}{2\pi i x_i} x_i^{km_i} \prod_{i \neq j} \left(1 - \frac{x_i}{x_j}\right) \\
\int \prod_i \frac{d\tilde{x}_i}{2\pi i \tilde{x}_i} \tilde{x}_i^{-k\tilde{m}_i} \prod_{i \neq j} \left(1 - \frac{\tilde{x}_i}{\tilde{x}_j}\right) \\
\prod_{i,j} \prod_a \left(\frac{\left(\frac{x_i}{x_j} y_a\right)^{1/2}}{1 - \frac{x_i}{x_j} y_a}\right)^{m_i - \tilde{m}_j - n_a + 1} \prod_{i,j} \prod_b \left(\frac{\left(\frac{\tilde{x}_i}{\tilde{x}_j} y_b\right)^{1/2}}{1 - \frac{\tilde{x}_i}{\tilde{x}_j} y_a}\right)^{\tilde{m}_j - m_i - n_b + 1}
$$

- The index can be evaluated from the Jeffrey-Kirwan residue
Eigenvalue distribution

- Single solution to the BAE up to permutations

Solution for $\Delta_a = \{0.3, 0.4, 0.5, 2\pi - 1.2\}$ and $N = 50$
Eigenvalue distribution

- Single solution to the BAE up to permutations

Solution for $\Delta_a = \{0.3, 0.4, 0.5, 2\pi - 1.2\}$ and $N = 50$

- Large-$N$ behavior

\[ \Re \log Z \sim f_0 N^{3/2} + f_1 N^{1/2} - \frac{1}{2} \log N + \cdots \]

- Subleading terms are difficult to extract analytically
  Tails in the distribution lead to complications
Massive IIA theory on $\text{AdS}_4 \times S^6$

- The dual field theory is an $\mathcal{N} = 2$ Chern-Simons-matter theory with $SU(N)_k$ gauge group and adjoint matter $X, Y, Z$ [Guarino, Jafferis and Varela, arXiv:1504.08009]

- Here the topologically twisted index is given by

$$Z(y_a, n_a) = \frac{(-1)^N}{N!} \sum_m \int \prod_i \frac{dx_i}{2\pi i x_i} x_i^{km_i} \prod_{i \neq j} \left(1 - \frac{x_i}{x_j}\right) \prod_{i,j} \prod_a \left(\frac{x_i}{x_j} y_a\right)^{1/2} \frac{1}{1 - \frac{x_i}{x_j} y_a}^{m_j - m_j + n_a + 1}$$

- Once again, the index is evaluated from the Jeffrey-Kirwan residue
Eigenvalue distribution

- Single solution to the BAE up to permutations

Solution for $\Delta_a = \{0.2, 0.7, 2\pi - 0.9\}$ and $N = 50$
Eigenvalue distribution

- Single solution to the BAE up to permutations

![Graph showing eigenvalue distribution with solution for \( \Delta_a = \{0.2, 0.7, 2\pi - 0.9\} \) and \( N = 50 \)]

- Large-\( N \) behavior

\[
\Re \log Z \sim f_0 N^{5/3} + f_1 N^{2/3} + f_2 N^{1/3} + f_3 \log N + \cdots
\]

- Can we understand the subleading behavior?
  
  No tails, but still have to deal with endpoints
Building blocks of the $S^2 \times T^2$ index

We now turn to the topologically twisted index on $S^2 \times T^2$ where $T^2$ is parametrized by $q = e^{2\pi i \tau}$

- Four-dimensional Yang-Mills theory on $S^2 \times T^2$
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  - Four-dimensional Yang-Mills theory on $S^2 \times T^2$

- Work in an $\mathcal{N} = 2$ language
  - Vector multiplets:
    \[
    Z_{\text{vector}} = (-1)^{2\rho(m)} \prod_{i \in G} \frac{dx_i}{2\pi i x_i} \eta(q)^2 \prod_{\alpha \in G} \left( \frac{\theta_1(x^\alpha, q)}{i\eta(q)} \right)
    \]
  - Chiral multiplets:
    \[
    Z_{\text{chiral}} = \prod_{\mu \in R} \left( \frac{i\eta(q)}{\theta_1(x^\mu y^{\mu_f}, q)} \right)^{\mu(m)+\mu_f(n)+1}
    \]
IIB on $AdS_5 \times S^5$

- The dual theory is $\mathcal{N} = 4$ SYM with $SU(N)$ gauge group.
  One vector and three chiral multiplets in the $\mathcal{N} = 1$ language.

- The topologically twisted index is [Hosseini, Nedelin and Zaffaroni, arXiv:1611.09374]

\[
Z(y_a, n_a) = \frac{1}{N!} \sum_m \int \prod_i \frac{dx_i}{2\pi i x_i} \eta(q)^2 \prod_{i \neq j} \left( \frac{\theta_1(\frac{x_i}{x_j}, q)}{i \eta(q)} \right) \prod_{i,j} \prod_a \left( \frac{i \eta(q)}{\theta_1(\frac{x_i}{x_j} y_a, q)} \right)^{m_i - m_j - n_a + 1}
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- After evaluating the Jeffrey-Kirwan residue

$$Z = A \sum_{l \in \text{BAEs}} \frac{1}{\det B} \prod_{i \neq j} \left[ \frac{\theta_1(\frac{x_i}{x_j}, q)}{i\eta(q)} \prod_a \left( \frac{i\eta(q)}{\theta_1(\frac{x_j}{x_j} y_a, q)} \right)^{1-n_a} \right]$$
Solving the BAE

The BAEs that we need to solve are

\[ 1 = e^{iB_i} \equiv e^{iv} \prod_j \prod_a \frac{\theta_1(e^{i(u_j-u_i+\Delta_a)}, q)}{\theta_1(e^{i(u_i-u_j+\Delta_a)}, q)} \]

How do we obtain the \( u_j \)'s?
Solving the BAE

- The BAEs that we need to solve are

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- How do we obtain the \( u_i \)'s?

Hosseini, Nedelin, Zaffaroni obtained \( u_j = \bar{u} + 2\pi \frac{\tau}{N} j \) in the high-temperature limit \( \beta \to 0^+ \) where \( \tau = i\beta/2\pi \)
Multiple solutions to the BAE

- Evenly distributed eigenvalues $\Rightarrow$ good solution for any $q$!
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- Evenly distributed eigenvalues $\implies$ good solution for any $q$!

$$\begin{align*}
\bar{u}_{jk} &= \bar{u} + \frac{2\pi}{m} j + \frac{2\pi}{n} k \\
\tau_{m,n} &= m \tau + \frac{r}{n}
\end{align*}$$

with $j = 0, 1, \ldots, m-1$ and $k = 0, 1, \ldots, n-1$
Multiple solutions to the BAE

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Multiple solutions to the BAE

- Evenly distributed eigenvalues \(\Rightarrow\) good solution for any \(q\)!

We find a family of exact solutions specified by \(\{m, n, r\}\) where \(N = mn\) and \(r = 0, 1, \ldots, n - 1\)

\[ u_{jk} = \bar{u} + 2\pi \frac{j + k\tilde{\tau}}{m} \quad \tilde{\tau} = \frac{m\tau + r}{n} \]

with \(j = 0, 1, \ldots, m - 1\) and \(k = 0, 1, \ldots, n - 1\)
The topologically twisted index

The topologically twisted index for $\mathcal{N} = 4$ SYM on $S^2 \times T^2$ can be written as $Z = \sum_{\{m,n,r\}} Z_{\{m,n,r\}}$ where

$$Z_{\{m,n,r\}}(\Delta_a, n_a) = \frac{j^{N-1}}{\det B_{\{m,n,r\}}} \prod_a \left[ \psi(\Delta_a, \tau) \left( \frac{m}{\psi(m\Delta_a, \bar{\tau})} \right)^N \right]^{1-n_a}$$

and

$$\psi(u, \tau) = \frac{\theta_1(u, \tau)}{\eta^3(\tau)} = \sqrt{\varphi_{-2,1}(u, \tau)}$$

Here $\varphi_{-2,1}$ is the unique weak Jacobi form of weight $-2$ and index $1$.
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The sum over sectors is crucial for modularity of the index.

Two modular parameters: $\tau : T^2$ and $\tilde{\tau} : T^2/\mathbb{Z}_m \times \mathbb{Z}_n$
The index as an elliptic genus

- The index computes the elliptic genus of the $\mathcal{N} = (0, 2)$ SCFT obtained by reducing on $S^2$
  
  Transforms under $SL(2, \mathbb{Z})$ as a weak Jacobi form of weight 0
The index as an elliptic genus

- The index computes the elliptic genus of the $\mathcal{N} = (0, 2)$ SCFT obtained by reducing on $S^2$
  - Transforms under $SL(2, \mathbb{Z})$ as a weak Jacobi form of weight 0
- Consider, for example, the case $N = 6$
The transformation $\mathcal{T} : \tau \rightarrow \tau + 1$
The transformation $S : \tau \rightarrow -1/\tau$
The high-temperature limit of the index

The high-temperature limit $\beta \to 0^+$ where $\tau = i\beta/2\pi$ can be obtained by performing a modular transformation $\tau \to -1/\tau$

We expect the index to be dominated by a single sector

This is the sector considered in Hosseini, Nedelin, Zaffaroni

$$\log Z(\Delta_a, n_a) \bigg|_{\beta \to 0^+} \sim \frac{\pi^2}{6\beta} c_r(\Delta_a, n_a)$$
What about the large-$N$ limit?

▶ In the Cardy limit, we expect $N^2$ behavior

$$\log Z \sim \frac{N^2}{\beta} \quad \text{ie} \quad c_r = \mathcal{O}(N^2)$$

This can be seen in the high-temperature limit of the topologically twisted index.
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This can be seen in the high-temperature limit of the topologically twisted index

- Is there a \(\log(N)\) correction?

  And if so, is it universal? Can it be reproduced in the AdS black string dual?
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This can be seen in the high-temperature limit of the topologically twisted index

- Is there a $\log(N)$ correction?
  And if so, is it universal? Can it be reproduced in the AdS black string dual?

- At finite temperature we expect modular covariance

\[
Z \sim N^2 \psi(\Delta_a, n_a, \tau)
\]

- Can we study the elliptic genus at large-$N$?
  And on the AdS side of the duality?
Summary

- We have explored the topologically twisted index for theories on $S^2 \times S^1$ and $S^2 \times T^2$
- **Main result**: There are multiple solutions to the BAE for the index on $S^2 \times T^2$
  - Needed for modular covariance of the index
  - But in the Cardy limit, only a single sector dominates
- Much remains to be understood in the precision counting of AdS black hole microstates

Junho Hong will say more in the Gong Show

Brian McPeak will talk about a separate project on the $D=6$ index
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- Brian McPeak will talk about a separate project on the $D = 6$ index.