The S-matrix Bootstrap for the 2d O(N) bosonic model

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Summary

- Introduction and Motivation

Define a field theory by a maximization problem in the space of allowed S-matrices

Simple example of S-matrix bootstrap: coupling as a functional

Maximizing linear functionals in convex spaces: vertices

Should apply to theories without continuous parameters (or parameters are fixed)
• **The 2d O(N) model from S-matrix bootstrap**

  The O(N) model (exact S matrices)

  The O(N) model from a convex maximization problem.

• **CDD factors and zero modes**

  More detailed structure of the space of theories.

• **Conclusions**
S-matrix bootstrap

The S-matrix satisfies constraints from analyticity, unitarity and crossing.

In the allowed space one can consider a functional and define a theory by the S-matrix that maximizes such functional.

A standard example is the coupling between a particle and its bound states (whose spectrum is assumed fixed). There is a maximum coupling because increasing the coupling further adds more bound states. Paulos, Penedones, Toledo, van Rees, Vieira

2d O(N) model has no bound states ....
Example: (Paulos, Penedones, Toledo, van Rees, Vieira) 2d theory, two species of particles of masses $m$ and $m_1 = 2m \sin \frac{\gamma}{16}$ where $\gamma$ is a real parameter.

Poles at $s_1 = m_1^2$ and $s_2 = 4m^2 - m_1^2$. ($t=4m^2-s$)

$$S = \frac{g}{z - ia} - \frac{g}{z + ia} + \hat{S}(z), \quad a = \frac{\cos \frac{\gamma}{8}}{1 + \sin \frac{\gamma}{8}},$$

$g$ : coupling constant. 

$|S(z)| \leq 1$ at the boundary of the disk.
\[ s = 4m^2 \cosh^2 \frac{\theta}{2} \]

\[ z = \frac{i - e^\theta}{i + e^\theta} \]
If, under those constraints, we maximize the parameter $g$ we obtain a simple result:

$$g_{max} = \frac{1 - a^4}{2a}, \quad \hat{S}_{max}(z) = i a^2,$$

$$S_{max}(z) = i \frac{1 + a^2 z^2}{z^2 + a^2} = \frac{\sinh \theta + i \sin \frac{\gamma}{8}}{\sinh \theta - i \sin \frac{\gamma}{8}}$$

Quite interestingly, $S_{max}(z)$ saturates the bounds and matches the S-matrix of two elementary particles associated with the scalar field in the sine-Gordon model. Thus, we see the S-matrix of a well-known model arising from such a simple maximization problem.
Maximizing a linear functional in a convex space is easy. There is only one local=global minimum and it is one of the vertices. Which one depends on the direction of the gradient.
The 2d O(N) model: N species of bosons w/ mass m

S-matrix

\[
S_{ab \rightarrow cd} = \left[ \delta_{ab}\delta_{cd} S_A(s) + \delta_{ac}\delta_{bd} S_T(s) + \delta_{ad}\delta_{bc} S_R(s) \right] \delta(p_1 - p_3)\delta(p_2 - p_4) \\
+ (p_3 \leftrightarrow p_4)(c \leftrightarrow d) \\
= \left( \frac{1}{N} \delta_{ab}\delta_{cd} S_I(s) + \frac{1}{2} (\delta_{ac}\delta_{bd} + \delta_{bc}\delta_{ad} - \frac{2}{N} \delta_{ab}\delta_{cd}) S_+(s) \\
+ \frac{1}{2} (\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}) S_-(s) \right) \delta(p_1 - p_3)\delta(p_2 - p_4) \\
+ (p_3 \leftrightarrow p_4)(c \leftrightarrow d)
\]
Exact result using YBE (Zamolodchikov-Zamolodchikov)

\[ Q(\theta) = \frac{\Gamma\left(\frac{\lambda-i\theta}{2\pi}\right) \Gamma\left(\frac{1}{2} - \frac{i\theta}{2\pi}\right)}{\Gamma\left(\frac{1}{2} + \frac{\lambda-i\theta}{2\pi}\right) \Gamma\left(-\frac{i\theta}{2\pi}\right)} \]

\[ S_T = Q(\theta)Q(i\pi - \theta) \]
\[ S_A = -\frac{i\lambda}{i\pi - \theta} S_T(s) \]
\[ S_R = -\frac{i\lambda}{\theta} S_T(s) \]

\[ S_I = NS_A + S_T + S_R, \quad S_\pm = S_T \pm S_R \]

\[ s = 4m^2 \cosh^2 \frac{\theta}{2} \]

\[ \lambda = \frac{2\pi}{N - 2} \]
General properties of the S matrix

**Unitarity**

\[ SS^\dagger = \mathbb{I} \]

Considering a subspace \( D \)

\[
\sum_{\beta \in D} \langle \psi_\alpha | S | \psi_\beta \rangle \langle \psi_\beta | S^\dagger | \psi_\alpha \rangle - \langle \psi_\alpha | \psi_\alpha \rangle = - \sum_{\beta \notin D} |\langle \psi_\alpha | S | \psi_\beta \rangle|^2 \leq 0
\]

\[ S_D S_D^\dagger \leq \mathbb{I} \quad \text{in a subspace or, equivalently,} \]

\[
\begin{pmatrix}
\mathbb{I} & S_D \\
S_D^\dagger & \mathbb{I}
\end{pmatrix} \succeq 0
\]

Defines a convex space of allowed \( S_D \) matrices
Here:

$$|S_I(\theta)|^2 \leq 1, \quad |S_+(\theta)|^2 \leq 1, \quad |S_-(\theta)|^2 \leq 1, \quad \theta \in \mathbb{R}$$

$$s > 4m^2$$

**Crossing**

$$S_a(i\pi - \theta) = \sum_b C_{ab} S_b(\theta), \quad a, b = I, +, -$$

$$C = \begin{pmatrix}
\frac{1}{N} & \frac{N}{2} + \frac{1}{2} - \frac{1}{N} & \frac{1}{2} - \frac{N}{2} \\
\frac{1}{N} & \frac{1}{2} - \frac{1}{N} & \frac{1}{2} \\
-\frac{1}{N} & \frac{1}{2} + \frac{1}{N} & \frac{1}{2}
\end{pmatrix}$$

With a linear constraint the space remains convex
In this space we consider a linear functional:

\[ F = S_1(\theta_0) - \alpha S_2(\theta_0) \quad \theta_0 \approx i \]

\[
S_1 = S_T = \frac{1}{2} (S_+ + S_-) \\
S_2 = \frac{1}{2} (S_R + S_A) = \frac{1}{2N} S_I + \frac{N - 2}{4N} S_+ - \frac{1}{4} S_- \\
S_3 = \frac{1}{2} (S_R - S_A) = -\frac{1}{2N} S_I + \frac{N + 2}{4N} S_+ - \frac{1}{4} S_- 
\]

Maximize \( F \) subject to the crossing constraints and unitarity bounds.

We do not use factorization (YBE).
N=5

maximization result vs integrable model on the physical line
Maximization result vs integrable model on the physical line

N=20
N=100

Maximization result vs integrable model on the physical line
Vertex: Absence of zero modes

Convex polygon

\[ V_B^C \xi_B \leq 0 \]
\[A = (\alpha, \sigma)\]

\[\text{Im} S_A = \sum_B K_{AB} \text{Re} S_B\]

\[|S_A|^2 = (\text{Re} S_A)^2 + \left(\sum_B K_{AB} \text{Re} S_B\right)^2 = \sum_{BC} \text{Re} S_B H_{BC}^A \text{Re} S_C \leq 1\]

\[\text{Re} S_A = \text{Re} \hat{S}_A + \xi_A\]

\[V_B^C \xi_B \leq 0, \quad V_B^C = \sum_C \text{Re} \hat{S}_A H_{AB}^C\]

\[H_{BC}^A = \delta_B^A \delta_C^A + K_{AB} K_{AC}\]

\[\tilde{V} = V^T V \quad \text{has no zero modes}\]
Iterative improvement of maximization functional

A general maximization functional is of the form

\[ F_w = \sum_A w_A \text{Re} S_A \]

Starting from

\[ F = S_1(\theta_0) - \alpha S_2(\theta_0) \]

we can refine \( w_A \) by taking an average of the normals:

\[ w_A = \sum_B V_{BA} \]

converges fast.
Analytical solution from numerics (example $N=8$)

\[ R_1 = \frac{S_1(\theta)}{S_-(\theta)}, \quad R_2 = \frac{S_+(\theta)}{S_-(\theta)} \]

\[ \tilde{R}_1 = \frac{\theta + i\pi}{\theta - i\pi}, \quad \tilde{R}_2 = \frac{\theta - 1.048i}{\theta + 1.048i} \]

From crossing, the position of the zeros, and the S-matrix can be found exactly.
CDD factors and zero modes (Castillejo-Dalitz-Dyson)

\[ F(\theta, \alpha) = \frac{\sinh \theta + i \sin \alpha}{\sinh \theta - i \sin \alpha} \]

\[ \tilde{S}_I(\theta) = \prod_i F(\theta, \alpha_i) S_I(\theta), \]
\[ \tilde{S}_+(\theta) = \prod_i F(\theta, \alpha_i) S_+(\theta), \]
\[ \tilde{S}_-(\theta) = \prod_i F(\theta, \alpha_i) S_-(\theta), \]

Have free parameters, it has 6 zero modes.
Conclusions

The S-matrix bootstrap provides a very interesting way to define a field theory as the maximum of a functional in the space of allowed S-matrices.

We argued that such space is convex and therefore, if a linear functional is maximized, the maximum is at the boundary and easily found by standard numerical methods.

Certain field theories have no free parameters and should be found at a vertex of such convex space.

This works well for the 2d O(N) model.